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If $a,b,c>0$, then prove that:

$$\sum_{cyc}\frac{\sqrt{b(b^2-bc+c^2)}+\sqrt{a(c^2-ca+a^2)}}{\sqrt{bc}+\sqrt{ac}}\geq\sqrt{3(a+b+c)}$$

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$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ *and analogs*$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with *area* F (*say*) *and* $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now},\forall x,y,z>0,\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt3}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac34$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac34\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt3}{2}\rightarrow(2)$$

$$\text{We have}:\frac{\sqrt{b(b^2-bc+c^2)}+\sqrt{a(c^2-ca+a^2)}}{\sqrt{bc}+\sqrt{ac}}+$$

$$\frac{\sqrt{b(a^2-ab+b^2)}+\sqrt{c(c^2-ca+a^2)}}{\sqrt{ab}+\sqrt{ac}}+\frac{\sqrt{a(a^2-ab+b^2)}+\sqrt{c(b^2-bc+c^2)}}{\sqrt{ab}+\sqrt{bc}}$$

$$=\frac{\sqrt{\frac{b^2-bc+c^2}{a}}+\sqrt{\frac{c^2-ca+a^2}{b}}}{\sqrt{\frac ca}+\sqrt{\frac cb}}+\frac{\sqrt{\frac{a^2-ab+b^2}{c}}+\sqrt{\frac{c^2-ca+a^2}{b}}}{\sqrt{\frac ac}+\sqrt{\frac ab}}+\frac{\sqrt{\frac{a^2-ab+b^2}{c}}+\sqrt{\frac{b^2-bc+c^2}{a}}}{\sqrt{\frac bc}+\sqrt{\frac ba}}$$

$$=\frac{\frac{1}{\sqrt a}}{\frac{1}{\sqrt b}+\frac{1}{\sqrt c}}\cdot\left(\sqrt{\frac{c^2-ca+a^2}{b}}+\sqrt{\frac{a^2-ab+b^2}{c}}\right)+\frac{\frac{1}{\sqrt b}}{\frac{1}{\sqrt c}+\frac{1}{\sqrt a}}\cdot\left(\sqrt{\frac{a^2-ab+b^2}{c}}+\sqrt{\frac{b^2-bc+c^2}{a}}\right)$$

$$+\frac{\frac{1}{\sqrt c}}{\frac{1}{\sqrt a}+\frac{1}{\sqrt b}}\cdot\left(\sqrt{\frac{b^2-bc+c^2}{a}}+\sqrt{\frac{c^2-ca+a^2}{b}}\right)$$

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$$= \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B)$$

$$\left(x = \frac{1}{\sqrt{a}}, y = \frac{1}{\sqrt{b}}, z = \frac{1}{\sqrt{c}}, A = \sqrt{\frac{b^2 - bc + c^2}{a}}, B = \sqrt{\frac{c^2 - ca + a^2}{b}}, C = \sqrt{\frac{a^2 - ab + b^2}{c}} \right)$$

$$= \frac{x}{y+z}.\sqrt{B+C}^2 + \frac{y}{z+x}.\sqrt{C+A}^2 + \frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB}.\frac{\sqrt{3}}{2}$$

$$= \sqrt{3}.\sqrt{\sum_{cyc} \left(\sqrt{\frac{b^2 - bc + c^2}{a}}.\sqrt{\frac{c^2 - ca + a^2}{b}} \right)} \overset{A-G}{\geq} \sqrt{3}.\sqrt{\sum_{cyc} \left(\sqrt{\frac{bc+}{a}}.\sqrt{\frac{ca}{b}} \right)}$$

$$= \sqrt{3(a+b+c)} \; \forall \, a, b, c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$