ROMANIAN MATHEMATICAL MAGAZINE

$\forall\, a,b,c>0$, prove that :

$$\sum_{cyc}\frac{\sqrt{\frac{b^n+c^n}{a^n+b^n}}+\sqrt{\frac{a^n+c^n}{a^n+b^n}}}{\frac{a+c}{a+b}.\sqrt{b^n+c^n}+\frac{b+c}{a+b}.\sqrt{a^n+c^n}}\ge\frac{3\sqrt{3}}{\sqrt{2(a^n+b^n+c^n)}}$$

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$\forall\, A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle
$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now},\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have}:\frac{\sqrt{\frac{b^n+c^n}{a^n+b^n}}+\sqrt{\frac{a^n+c^n}{a^n+b^n}}}{\frac{a+c}{a+b}.\sqrt{b^n+c^n}+\frac{b+c}{a+b}.\sqrt{a^n+c^n}}+\frac{\sqrt{\frac{a^n+b^n}{b^n+c^n}}+\sqrt{\frac{a^n+c^n}{b^n+c^n}}}{\frac{a+c}{b+c}.\sqrt{a^n+b^n}+\frac{a+b}{b+c}.\sqrt{a^n+c^n}}$$

$$+\frac{\sqrt{\frac{a^n+b^n}{a^n+c^n}}+\sqrt{\frac{b^n+c^n}{a^n+c^n}}}{\frac{b+c}{a+c}.\sqrt{a^n+b^n}+\frac{a+b}{a+c}.\sqrt{b^n+c^n}}$$

$$=\frac{\left(\frac{a+b}{\sqrt{a^n+b^n}}\right)\left(\sqrt{b^n+c^n}+\sqrt{c^n+a^n}\right)}{\frac{b+c}{\sqrt{b^n+c^n}}.\sqrt{(b^n+c^n)(c^n+a^n)}+\frac{c+a}{\sqrt{c^n+a^n}}.\sqrt{(c^n+a^n)(b^n+c^n)}}$$

$$+\frac{\left(\frac{b+c}{\sqrt{b^n+c^n}}\right)\left(\sqrt{c^n+a^n}+\sqrt{a^n+b^n}\right)}{\frac{c+a}{\sqrt{c^n+a^n}}.\sqrt{(c^n+a^n)(a^n+b^n)}+\frac{a+b}{\sqrt{a^n+b^n}}.\sqrt{(a^n+b^n)(c^n+a^n)}}$$

$$+\frac{\left(\frac{c+a}{\sqrt{c^n+a^n}}\right)\left(\sqrt{a^n+b^n}+\sqrt{b^n+c^n}\right)}{\frac{a+b}{\sqrt{a^n+b^n}}.\sqrt{(a^n+b^n)(b^n+c^n)}+\frac{b+c}{\sqrt{b^n+c^n}}.\sqrt{(b^n+c^n)(a^n+b^n)}}$$

$$= \frac{\frac{a+b}{\sqrt{a^n+b^n}}}{\frac{b+c}{\sqrt{b^n+c^n}} + \frac{c+a}{\sqrt{c^n+a^n}}} \cdot \left(\frac{1}{\sqrt{b^n+c^n}} + \frac{1}{\sqrt{c^n+a^n}}\right)$$

$$+ \frac{\frac{b+c}{\sqrt{b^n+c^n}}}{\frac{c+a}{\sqrt{c^n+a^n}} + \frac{a+b}{\sqrt{a^n+b^n}}} \cdot \left(\frac{1}{\sqrt{c^n+a^n}} + \frac{1}{\sqrt{a^n+b^n}}\right)$$

$$+ \frac{\frac{c+a}{\sqrt{c^n+a^n}}}{\frac{a+b}{\sqrt{a^n+b^n}} + \frac{b+c}{\sqrt{b^n+c^n}}} \cdot \left(\frac{1}{\sqrt{a^n+b^n}} + \frac{1}{\sqrt{b^n+c^n}}\right)$$

$$= \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{l} x = \frac{a+b}{\sqrt{a^n+b^n}}, y = \frac{b+c}{\sqrt{b^n+c^n}}, z = \frac{c+a}{\sqrt{c^n+a^n}}, \\ A = \frac{1}{\sqrt{a^n+b^n}}, B = \frac{1}{\sqrt{b^n+c^n}}, C = \frac{1}{\sqrt{c^n+a^n}} \end{array}\right)$$

$$= \frac{x}{y+z}.\sqrt{B+C}^2 + \frac{y}{z+x}.\sqrt{C+A}^2 + \frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{\text{cyc}} \frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{\text{cyc}} AB}.\frac{\sqrt{3}}{2} = \sqrt{3\sum_{\text{cyc}} \frac{1}{\sqrt{(a^n+b^n)(b^n+c^n)}}}$$

$$\overset{\text{Bergstrom}}{\geq} \sqrt{\frac{27}{\sum_{\text{cyc}} \sqrt{(a^n+b^n)(b^n+c^n)}}} \overset{\text{CBS}}{\geq} \sqrt{\frac{27}{\sqrt{\left(\sum_{\text{cyc}}(a^n+b^n)\right)\left(\sum_{\text{cyc}}(b^n+c^n)\right)}}}$$

$$= \frac{3\sqrt{3}}{\sqrt{2(a^n+b^n+c^n)}} \ \forall\, a,b,c > 0, '' ='' \text{ iff } a=b=c \text{ (QED)}$$