ROMANIAN MATHEMATICAL MAGAZINE

$\forall\, a,b,c>0, \textit{prove that}:$

$$\sum_{cyc}\frac{\sqrt{\frac{b^2+c^2}{a^2+b^2}}+\sqrt{\frac{a^2+c^2}{a^2+b^2}}}{\frac{a+c}{a+b}.\sqrt{b^2+c^2}+\frac{b+c}{a+b}.\sqrt{a^2+c^2}}\ge\frac{3\sqrt3}{\sqrt{2(a^2+b^2+c^2)}}$$

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$\forall\, A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

Now, $\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt3}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac34$

Via Bergstrom, LHS of $(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$

$$\overset{?}{\ge}\frac34\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt3}{2}\rightarrow(2)$$

We have :

$$\frac{\sqrt{\frac{b^2+c^2}{a^2+b^2}}+\sqrt{\frac{a^2+c^2}{a^2+b^2}}}{\frac{a+c}{a+b}.\sqrt{b^2+c^2}+\frac{b+c}{a+b}.\sqrt{a^2+c^2}}+\frac{\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\sqrt{\frac{a^2+c^2}{b^2+c^2}}}{\frac{a+c}{b+c}.\sqrt{a^2+b^2}+\frac{a+b}{b+c}.\sqrt{a^2+c^2}}$$

$$+\frac{\sqrt{\frac{a^2+b^2}{a^2+c^2}}+\sqrt{\frac{b^2+c^2}{a^2+c^2}}}{\frac{b+c}{a+c}.\sqrt{a^2+b^2}+\frac{a+b}{a+c}.\sqrt{b^2+c^2}}$$

$$=\frac{\left(\frac{a+b}{\sqrt{a^2+b^2}}\right)\left(\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)}{\frac{b+c}{\sqrt{b^2+c^2}}.\sqrt{(b^2+c^2)(c^2+a^2)}+\frac{c+a}{\sqrt{c^2+a^2}}.\sqrt{(c^2+a^2)(b^2+c^2)}}$$

$$+\frac{\left(\frac{b+c}{\sqrt{b^2+c^2}}\right)\left(\sqrt{c^2+a^2}+\sqrt{a^2+b^2}\right)}{\frac{c+a}{\sqrt{c^2+a^2}}.\sqrt{(c^2+a^2)(a^2+b^2)}+\frac{a+b}{\sqrt{a^2+b^2}}.\sqrt{(a^2+b^2)(c^2+a^2)}}$$

$$+\frac{\left(\frac{c+a}{\sqrt{c^2+a^2}}\right)\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}\right)}{\frac{a+b}{\sqrt{a^2+b^2}}.\sqrt{(a^2+b^2)(b^2+c^2)}+\frac{b+c}{\sqrt{b^2+c^2}}.\sqrt{(b^2+c^2)(a^2+b^2)}}$$

$$=\frac{\frac{a+b}{\sqrt{a^2+b^2}}}{\frac{b+c}{\sqrt{b^2+c^2}}+\frac{c+a}{\sqrt{c^2+a^2}}}\cdot\left(\frac{1}{\sqrt{b^2+c^2}}+\frac{1}{\sqrt{c^2+a^2}}\right)+\frac{\frac{b+c}{\sqrt{b^2+c^2}}}{\frac{c+a}{\sqrt{c^2+a^2}}+\frac{a+b}{\sqrt{a^2+b^2}}}\cdot\left(\frac{1}{\sqrt{c^2+a^2}}+\frac{1}{\sqrt{a^2+b^2}}\right)+$$

$$+\frac{\frac{c+a}{\sqrt{c^2+a^2}}}{\frac{a+b}{\sqrt{a^2+b^2}}+\frac{b+c}{\sqrt{b^2+c^2}}}\cdot\left(\frac{1}{\sqrt{a^2+b^2}}+\frac{1}{\sqrt{b^2+c^2}}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{l} x=\frac{a+b}{\sqrt{a^2+b^2}},y=\frac{b+c}{\sqrt{b^2+c^2}},z=\frac{c+a}{\sqrt{c^2+a^2}}, \\ A=\frac{1}{\sqrt{a^2+b^2}},B=\frac{1}{\sqrt{b^2+c^2}},C=\frac{1}{\sqrt{c^2+a^2}} \end{array}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\frac{1}{\sqrt{(a^2+b^2)(b^2+c^2)}}}$$

$$\overset{\text{Bergstrom}}{\geq}\sqrt{\frac{27}{\sum_{cyc}\sqrt{(a^2+b^2)(b^2+c^2)}}}\overset{\text{CBS}}{\geq}\sqrt{\frac{27}{\sqrt{\left(\sum_{cyc}(a^2+b^2)\right)\left(\sum_{cyc}(b^2+c^2)\right)}}}$$

$$=\frac{3\sqrt{3}}{\sqrt{2(a^2+b^2+c^2)}}\ \forall\, a,b,c>0,''='' \text{ iff } a=b=c\ (\text{QED})$$