ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $n\in\mathbb{N}$ with $n\geq 2$, then prove that :

$$\sum_{cyc}\frac{(a+c).\sqrt{\frac{b^n+c^n}{a^n+b^n}}+(b+c).\sqrt{\frac{a^n+c^n}{a^n+b^n}}}{\sqrt{b^n+c^n}+\sqrt{a^n+c^n}}\geq\frac{3\sqrt{6}.\sqrt[3]{abc}}{\sqrt{a^n+b^n+c^n}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall\, A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now},\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have}:\frac{(a+c).\sqrt{\frac{b^n+c^n}{a^n+b^n}}+(b+c).\sqrt{\frac{a^n+c^n}{a^n+b^n}}}{\sqrt{b^n+c^n}+\sqrt{a^n+c^n}}+\frac{(a+c).\sqrt{\frac{a^n+b^n}{b^n+c^n}}+(a+b).\sqrt{\frac{a^n+c^n}{b^n+c^n}}}{\sqrt{a^n+b^n}+\sqrt{a^n+c^n}}$$

$$+\frac{(b+c).\sqrt{\frac{a^n+b^n}{a^n+c^n}}+(a+b).\sqrt{\frac{b^n+c^n}{a^n+c^n}}}{\sqrt{a^n+b^n}+\sqrt{b^n+c^n}}$$

$$=\frac{\frac{\left(\frac{a+c}{\sqrt{a^n+c^n}}\right)}{\sqrt{a^n+b^n}}+\frac{\left(\frac{b+c}{\sqrt{b^n+c^n}}\right)}{\sqrt{a^n+b^n}}}{\frac{1}{\sqrt{a^n+c^n}}+\frac{1}{\sqrt{b^n+c^n}}}+\frac{\frac{\left(\frac{a+c}{\sqrt{a^n+c^n}}\right)}{\sqrt{b^n+c^n}}+\frac{\left(\frac{a+b}{\sqrt{a^n+b^n}}\right)}{\sqrt{b^n+c^n}}}{\frac{1}{\sqrt{a^n+c^n}}+\frac{1}{\sqrt{a^n+b^n}}}+\frac{\frac{\left(\frac{b+c}{\sqrt{b^n+c^n}}\right)}{\sqrt{a^n+c^n}}+\frac{\left(\frac{a+b}{\sqrt{a^n+b^n}}\right)}{\sqrt{a^n+c^n}}}{\frac{1}{\sqrt{b^n+c^n}}+\frac{1}{\sqrt{a^n+b^n}}}$$

$$=\frac{\frac{1}{\sqrt{a^n+b^n}}}{\frac{1}{\sqrt{b^n+c^n}}+\frac{1}{\sqrt{c^n+a^n}}}\cdot\left(\frac{b+c}{\sqrt{b^n+c^n}}+\frac{c+a}{\sqrt{c^n+a^n}}\right)+$$

$$\frac{\frac{1}{\sqrt{b^n+c^n}}}{\frac{1}{\sqrt{c^n+a^n}}+\frac{1}{\sqrt{a^n+b^n}}}\cdot\left(\frac{c+a}{\sqrt{c^n+a^n}}+\frac{a+b}{\sqrt{a^n+b^n}}\right)+$$

$$\frac{\frac{1}{\sqrt{c^n+a^n}}}{\frac{1}{\sqrt{a^n+b^n}}+\frac{1}{\sqrt{b^n+c^n}}}\cdot\left(\frac{a+b}{\sqrt{a^n+b^n}}+\frac{b+c}{\sqrt{b^n+c^n}}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{c} x=\frac{1}{\sqrt{a^n+b^n}}, y=\frac{1}{\sqrt{b^n+c^n}}, z=\frac{1}{\sqrt{c^n+a^n}}, \\ A=\frac{a+b}{\sqrt{a^n+b^n}}, B=\frac{b+c}{\sqrt{b^n+c^n}}, C=\frac{c+a}{\sqrt{c^n+a^n}} \end{array}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}} = \sqrt{3\sum_{cyc}\left(\frac{a+b}{\sqrt{a^n+b^n}}.\frac{b+c}{\sqrt{b^n+c^n}}\right)}$$

$$\overset{A-G}{\geq} 3.\sqrt[6]{\frac{(a+b)^2(b+c)^2(c+a)^2}{(a^n+b^n)(b^n+c^n)(c^n+a^n)}} \overset{\substack{\text{Cesaro}\\ \text{and}\\ A-G}}{\geq} 3.\frac{\sqrt[6]{64a^2b^2c^2}}{\sqrt{\frac{\sum_{cyc}(b^n+c^n)}{3}}} = \frac{6\sqrt{3}.\sqrt[3]{abc}}{\sqrt{2(a^n+b^n+c^n)}}$$

$$=\frac{3\sqrt{6}.\sqrt[3]{abc}}{\sqrt{a^n+b^n+c^n}} \ \forall\, a,b,c>0, \text{"="} \text{ iff } a=b=c \text{ (QED)}$$