ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ then:

$$\sum_{cyc}\frac{1}{a+b}\left(\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\sqrt{\frac{a^2+b^2}{a^2+c^2}}\right)\ge\frac{3\sqrt{3}}{\sqrt{a^2+b^2+c^2}}$$

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Solution by Tapas Das-India

$$(a+b+c)^2\overset{CBS}{\le}3(a^2+b^2+c^2)\ \ (1)$$

$$\frac{1}{a+b}\left(\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\sqrt{\frac{a^2+b^2}{a^2+c^2}}\right)\overset{AM-GM}{\ge}\frac{2}{a+b}\frac{\sqrt{a^2+b^2}}{\sqrt[4]{(b^2+c^2)(a^2+c^2)}},$$

similarly

$$\frac{1}{b+c}\left(\sqrt{\frac{b^2+c^2}{a^2+c^2}}+\sqrt{\frac{b^2+c^2}{a^2+b^2}}\right)\ge\frac{2}{c+b}\frac{\sqrt{c^2+b^2}}{\sqrt[4]{(b^2+a^2)(a^2+c^2)}},$$

and

$$\frac{1}{a+c}\left(\sqrt{\frac{a^2+c^2}{a^2+b^2}}+\sqrt{\frac{a^2+c^2}{b^2+c^2}}\right)\ge\frac{2}{a+c}\frac{\sqrt{a^2+c^2}}{\sqrt[4]{(b^2+c^2)(a^2+b^2)}}$$

$$\frac{1}{a+b}\left(\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\sqrt{\frac{a^2+b^2}{a^2+c^2}}\right)+\frac{1}{b+c}\left(\sqrt{\frac{b^2+c^2}{a^2+c^2}}+\sqrt{\frac{b^2+c^2}{a^2+b^2}}\right)$$
$$+\frac{1}{a+c}\left(\sqrt{\frac{a^2+c^2}{a^2+b^2}}+\sqrt{\frac{a^2+c^2}{b^2+c^2}}\right)=$$
$$\sum\frac{1}{a+b}\left(\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\sqrt{\frac{a^2+b^2}{a^2+c^2}}\right)\ge\sum\frac{2}{a+b}\frac{\sqrt{a^2+b^2}}{\sqrt[4]{(b^2+c^2)(a^2+c^2)}}\overset{am-gm}{\ge}$$
$$6\left(\frac{1}{\prod(a+b)}\right)^{\frac{1}{3}}\overset{am-gm}{\ge}\frac{18}{2\sum a}\overset{(1)}{\ge}\frac{9}{\left(3(a^2+b^2+c^2)\right)^{\frac{1}{2}}}=\frac{3\sqrt{3}}{\sqrt{a^2+b^2+c^2}}$$

Equality for a=b=c

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