ROMANIAN MATHEMATICAL MAGAZINE

If $a,\mathrm{b},\mathrm{c}>0,$ then prove that :

$$\sum_{cyc}\frac{\frac{a+c}{a+b}.\sqrt{b^2+c^2}+\frac{b+c}{a+b}.\sqrt{a^2+c^2}}{b+c+a+c}\geq\frac{3}{\sqrt{2}}$$

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$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle ($\because$ (A + B) + (B + C) > (C + A) and analogs) $\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

Now, $\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$

Via Bergstrom, LHS of $(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\ \therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

We have : $\frac{\frac{a+c}{a+b}.\sqrt{b^2+c^2}+\frac{b+c}{a+b}.\sqrt{a^2+c^2}}{b+c+a+c}+\frac{\frac{a+c}{b+c}.\sqrt{a^2+b^2}+\frac{a+b}{b+c}.\sqrt{a^2+c^2}}{a+b+a+c}$

$$+\frac{\frac{b+c}{a+c}.\sqrt{a^2+b^2}+\frac{a+b}{a+c}.\sqrt{b^2+c^2}}{a+b+b+c}$$

$$=\frac{\frac{1}{(a+b)(b+c)}.\sqrt{b^2+c^2}+\frac{1}{(a+b)(a+c)}.\sqrt{a^2+c^2}}{\frac{1}{a+c}+\frac{1}{b+c}}+$$

$$\frac{\frac{a+c}{(b+c)(a+b)}.\sqrt{a^2+b^2}+\frac{a+b}{(b+c)(a+c)}.\sqrt{a^2+c^2}}{\frac{1}{a+c}+\frac{1}{a+b}}+$$

$$\frac{\frac{1}{(a+c)(a+b)}.\sqrt{a^2+b^2}+\frac{1}{(a+c)(b+c)}.\sqrt{b^2+c^2}}{\frac{1}{b+c}+\frac{1}{a+b}}$$

$$=\frac{\frac{1}{a+b}}{\frac{1}{b+c}+\frac{1}{c+a}}.\left(\frac{\sqrt{b^2+c^2}}{b+c}+\frac{\sqrt{c^2+a^2}}{c+a}\right)+\frac{\frac{1}{b+c}}{\frac{1}{c+a}+\frac{1}{a+b}}.\left(\frac{\sqrt{c^2+a^2}}{c+a}+\frac{\sqrt{a^2+b^2}}{a+b}\right)$$

$$+\frac{\frac{1}{c+a}}{\frac{1}{a+b}+\frac{1}{b+c}}.\left(\frac{\sqrt{a^2+b^2}}{a+b}+\frac{\sqrt{b^2+c^2}}{b+c}\right)$$

$$= \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B)$$

$$\left(x = \frac{1}{a+b}, y = \frac{1}{b+c}, z = \frac{1}{c+a}, A = \frac{\sqrt{a^2+b^2}}{a+b}, B = \frac{\sqrt{b^2+c^2}}{b+c}, C = \frac{\sqrt{c^2+a^2}}{c+a}\right)$$

$$= \frac{x}{y+z}.\sqrt{B+C}^2 + \frac{y}{z+x}.\sqrt{C+A}^2 + \frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB}.\frac{\sqrt{3}}{2} = \sqrt{3\sum_{cyc}\left(\frac{\sqrt{a^2+b^2}}{a+b}.\frac{\sqrt{b^2+c^2}}{b+c}\right)}$$

$$\geq \sqrt{3\sum_{cyc}\left(\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}\right)} = \sqrt{\frac{9}{2}} \therefore \frac{\frac{a+c}{a+b}.\sqrt{b^2+c^2} + \frac{b+c}{a+b}.\sqrt{a^2+c^2}}{b+c+a+c} +$$

$$\frac{\frac{a+c}{b+c}.\sqrt{a^2+b^2} + \frac{a+b}{b+c}.\sqrt{a^2+c^2}}{a+b+a+c} + \frac{\frac{b+c}{a+c}.\sqrt{a^2+b^2} + \frac{a+b}{a+c}.\sqrt{b^2+c^2}}{a+b+b+c} \geq \frac{3}{\sqrt{2}}$$

$$\forall\, a,b,c > 0, \text{''} = \text{''} \text{ iff } a = b = c \text{ (QED)}$$