ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $n \in \mathbb{N}^* - \{1\}$, then prove that :

$$\sum_{cyc} \frac{\frac{a+c}{a+b}.\sqrt{b^n + c^n} + \frac{b+c}{a+b}.\sqrt{a^n + c^n}}{b + c + a + c} \geq \frac{3}{\sqrt{2}}(abc)^{\frac{n-2}{6}}$$

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$\forall A, B, C > 0, (A+B), (B+C), (C+A)$ form sides of a triangle
$(\because (A+B)+(B+C) > (C+A)$ and analogs$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc} AB + B^2\right) - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB$$

$$= 6\sum_{cyc} AB + 2\sum_{cyc} A^2 - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB \Rightarrow 4F = 2\sqrt{\sum_{cyc} AB} \to (1)$$

Now, $\forall x, y, z > 0$, $\sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc} \frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$

Via Bergstrom, LHS of $(*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

We have : $\frac{\frac{a+c}{a+b}.\sqrt{b^n + c^n} + \frac{b+c}{a+b}.\sqrt{a^n + c^n}}{b + c + a + c} + \frac{\frac{a+c}{b+c}.\sqrt{a^n + b^n} + \frac{a+b}{b+c}.\sqrt{a^n + c^n}}{a + b + a + c}$

$$+ \frac{\frac{b+c}{a+c}.\sqrt{a^n + b^n} + \frac{a+b}{a+c}.\sqrt{b^n + c^n}}{a + b + b + c}$$

$$= \frac{\frac{1}{(a+b)(b+c)}.\sqrt{b^n + c^n} + \frac{1}{(a+b)(a+c)}.\sqrt{a^n + c^n}}{\frac{1}{a+c} + \frac{1}{b+c}} +$$

$$\frac{\frac{a+c}{(b+c)(a+b)}.\sqrt{a^n + b^n} + \frac{a+b}{(b+c)(a+c)}.\sqrt{a^n + c^n}}{\frac{1}{a+c} + \frac{1}{a+b}} +$$

$$\frac{\frac{1}{(a+c)(a+b)}.\sqrt{a^n + b^n} + \frac{1}{(a+c)(b+c)}.\sqrt{b^n + c^n}}{\frac{1}{b+c} + \frac{1}{a+b}}$$

$$= \frac{\frac{1}{a+b}}{\frac{1}{b+c} + \frac{1}{c+a}}.\left(\frac{\sqrt{b^n + c^n}}{b+c} + \frac{\sqrt{c^n + a^n}}{c+a}\right) + \frac{\frac{1}{b+c}}{\frac{1}{c+a} + \frac{1}{a+b}}.\left(\frac{\sqrt{c^n + a^n}}{c+a} + \frac{\sqrt{a^n + b^n}}{a+b}\right)$$

$$+\frac{\frac{1}{c+a}}{\frac{1}{a+b}+\frac{1}{b+c}}\cdot\left(\frac{\sqrt{a^n+b^n}}{a+b}+\frac{\sqrt{b^n+c^n}}{b+c}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{1}{a+b},y=\frac{1}{b+c},z=\frac{1}{c+a},A=\frac{\sqrt{a^n+b^n}}{a+b},B=\frac{\sqrt{b^n+c^n}}{b+c},C=\frac{\sqrt{c^n+a^n}}{c+a}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{\sqrt{a^n+b^n}}{a+b}.\frac{\sqrt{b^n+c^n}}{b+c}\right)}$$

$$\overset{A-G}{\geq}3.\sqrt{\sqrt[3]{\prod_{cyc}\left(\frac{\sqrt{a^n+b^n}}{a+b}.\frac{\sqrt{b^n+c^n}}{b+c}\right)}}\overset{\text{Cesaro}}{\geq}3.\sqrt[6]{\frac{(a^n+b^n)(b^n+c^n)(c^n+a^n)}{(a+b)^2(b+c)^2(c+a)^2}}$$

$$\overset{\substack{\text{Chebyshev}\\ \text{and}\\ \text{Reverse CBS}}}{\geq}3.\sqrt[6]{\frac{\frac{(a^2+b^2)(a^{n-2}+b^{n-2})}{2}.\frac{(b^2+c^2)(b^{n-2}+c^{n-2})}{2}.\frac{(c^2+a^2)(c^{n-2}+a^{n-2})}{2}}{(2a^2+2b^2)(2b^2+2c^2)(2c^2+2a^2)}}\overset{\text{Cesaro}}{\geq}3.\sqrt[6]{\frac{8(abc)^{n-2}}{64}}$$

$$\therefore\frac{\frac{a+c}{a+b}.\sqrt{b^n+c^n}+\frac{b+c}{a+b}.\sqrt{a^n+c^n}}{b+c+a+c}+\frac{\frac{a+c}{b+c}.\sqrt{a^n+b^n}+\frac{a+b}{b+c}.\sqrt{a^n+c^n}}{a+b+a+c}$$

$$+\frac{\frac{b+c}{a+c}.\sqrt{a^n+b^n}+\frac{a+b}{a+c}.\sqrt{b^n+c^n}}{a+b+b+c}\geq\frac{3}{\sqrt{2}}(abc)^{\frac{n-2}{6}}\ \forall\ a,b,c>0\ \middle|\ n\in\mathbb{N}^*-\{1\},$$

$$''='' \text{ iff } a=b=c\ (\text{QED})$$