ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then prove that :

$$\sum_{cyc}\frac{\frac{b+c}{a+b}+\frac{c+a}{a+b}}{(b+c).\sqrt{\frac{c^2+a^2}{a^2+b^2}}+(c+a).\sqrt{\frac{b^2+c^2}{a^2+b^2}}}\geq\frac{9}{2(a+b+c)}$$

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$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now},\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have}:\frac{\frac{b+c}{a+b}+\frac{c+a}{a+b}}{(b+c).\sqrt{\frac{c^2+a^2}{a^2+b^2}}+(c+a).\sqrt{\frac{b^2+c^2}{a^2+b^2}}}+\frac{\frac{c+a}{b+c}+\frac{a+b}{b+c}}{(c+a).\sqrt{\frac{a^2+b^2}{b^2+c^2}}+(a+b).\sqrt{\frac{c^2+a^2}{b^2+c^2}}}$$

$$+\frac{\frac{a+b}{a+c}+\frac{b+c}{a+c}}{(a+b).\sqrt{\frac{b^2+c^2}{a^2+c^2}}+(b+c).\sqrt{\frac{a^2+b^2}{a^2+c^2}}}$$

$$=\frac{\frac{1}{(a+b)(c+a)}+\frac{1}{(a+b)(b+c)}}{\frac{1}{c+a}.\sqrt{\frac{c^2+a^2}{a^2+b^2}}+\frac{1}{b+c}.\sqrt{\frac{b^2+c^2}{a^2+b^2}}}+\frac{\frac{1}{(b+c)(a+b)}+\frac{1}{(b+c)(c+a)}}{\frac{1}{a+b}.\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\frac{1}{c+a}.\sqrt{\frac{c^2+a^2}{b^2+c^2}}}$$

$$+\frac{\frac{1}{(c+a)(b+c)}+\frac{1}{(c+a)(a+b)}}{\frac{1}{b+c}.\sqrt{\frac{b^2+c^2}{a^2+c^2}}+\frac{1}{a+b}.\sqrt{\frac{a^2+b^2}{a^2+c^2}}}$$

$$=\frac{\frac{\sqrt{a^2+b^2}}{a+b}}{\frac{\sqrt{b^2+c^2}}{b+c}+\frac{\sqrt{c^2+a^2}}{c+a}}.\left(\frac{1}{b+c}+\frac{1}{c+a}\right)+\frac{\frac{\sqrt{b^2+c^2}}{b+c}}{\frac{\sqrt{c^2+a^2}}{c+a}+\frac{\sqrt{a^2+b^2}}{a+b}}.\left(\frac{1}{c+a}+\frac{1}{a+b}\right)+$$

$$\frac{\frac{\sqrt{c^2+a^2}}{c+a}}{\frac{\sqrt{a^2+b^2}}{a+b}+\frac{\sqrt{b^2+c^2}}{b+c}}\cdot\left(\frac{1}{a+b}+\frac{1}{b+c}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{\sqrt{a^2+b^2}}{a+b},y=\frac{\sqrt{b^2+c^2}}{b+c},z=\frac{\sqrt{c^2+a^2}}{c+a},A=\frac{1}{a+b},B=\frac{1}{b+c},C=\frac{1}{c+a}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}=\sqrt{3\sum_{cyc}\left(\frac{1}{a+b}.\frac{1}{b+c}\right)}$$

$$\overset{\text{Bergstrom}}{\geq}3\sqrt{3}.\sqrt{\frac{1}{\sum_{cyc}((a+b)(b+c))}}=\frac{3\sqrt{3}}{\sqrt{\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab}}\geq$$

$$\frac{3\sqrt{3}}{\sqrt{\left(\sum_{cyc}a\right)^2+\frac{\left(\sum_{cyc}a\right)^2}{3}}}=\frac{9}{\sqrt{4\left(\sum_{cyc}a\right)^2}}\therefore\sum_{cyc}\frac{\frac{b+c}{a+b}+\frac{c+a}{a+b}}{(b+c).\sqrt{\frac{c^2+a^2}{a^2+b^2}}+(c+a).\sqrt{\frac{b^2+c^2}{a^2+b^2}}}$$

$$\geq\frac{9}{2(a+b+c)}\ \forall\, a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$