ROMANIAN MATHEMATICAL MAGAZINE

If $a,\mathrm{b},\mathrm{c}>0$ and $\mathrm{m},\mathrm{n}\in\mathbb{N}$, then prove that :

$$\sum_{cyc}\frac{\left(\frac{b+c}{a+b}\right)^m+\left(\frac{c+a}{a+b}\right)^m}{(b+c)^m.\sqrt{\frac{c^n+a^n}{a^n+b^n}}+(c+a)^m.\sqrt{\frac{b^n+c^n}{a^n+b^n}}}\geq\frac{9}{2^m.(a^m+b^m+c^m)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C >0, (A + B), (B + C), (C + A) form sides of a triangle
$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now, }\forall\,x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{\left(\frac{b+c}{a+b}\right)^m+\left(\frac{c+a}{a+b}\right)^m}{(b+c)^m.\sqrt{\frac{c^n+a^n}{a^n+b^n}}+(c+a)^m.\sqrt{\frac{b^n+c^n}{a^n+b^n}}}+$$

$$\frac{\left(\frac{c+a}{b+c}\right)^m+\left(\frac{a+b}{b+c}\right)^m}{(c+a)^m.\sqrt{\frac{a^n+b^n}{b^n+c^n}}+(a+b)^m.\sqrt{\frac{c^n+a^n}{b^n+c^n}}}+\frac{\left(\frac{a+b}{a+c}\right)^m+\left(\frac{b+c}{a+c}\right)^m}{(a+b)^m.\sqrt{\frac{b^n+c^n}{a^n+c^n}}+(b+c)^m.\sqrt{\frac{a^n+b^n}{a^n+c^n}}}$$

$$=\frac{\left(\frac{1}{(a+b)^m(c+a)^m}+\frac{1}{(a+b)^m(b+c)^m}\right).\sqrt{a^n+b^n}}{\frac{1}{(c+a)^m}.\sqrt{c^n+a^n}+\frac{1}{(b+c)^m}.\sqrt{b^n+c^n}}+$$

$$\frac{\left(\frac{1}{(b+c)^m(a+b)^m}+\frac{1}{(b+c)^m(c+a)^m}\right).\sqrt{b^n+c^n}}{\frac{1}{(a+b)^m}.\sqrt{a^n+b^n}+\frac{1}{(c+a)^m}.\sqrt{c^n+a^n}}+\frac{\left(\frac{1}{(c+a)^m(b+c)^m}+\frac{1}{(c+a)^m(a+b)^m}\right).\sqrt{c^n+a^n}}{\frac{1}{(b+c)^m}.\sqrt{b^n+c^n}+\frac{1}{(a+b)^m}.\sqrt{a^n+b^n}}$$

$$=\frac{\frac{\sqrt{a^n+b^n}}{(a+b)^m}.\left(\frac{1}{(b+c)^m}+\frac{1}{(c+a)^m}\right)}{\frac{\sqrt{b^n+c^n}}{(b+c)^m}+\frac{\sqrt{c^n+a^n}}{(c+a)^m}}+\frac{\frac{\sqrt{b^n+c^n}}{(b+c)^m}.\left(\frac{1}{(c+a)^m}+\frac{1}{(a+b)^m}\right)}{\frac{\sqrt{c^n+a^n}}{(c+a)^m}+\frac{\sqrt{a^n+b^n}}{(a+b)^m}}+\frac{\frac{\sqrt{c^n+a^n}}{(c+a)^m}}{\frac{\sqrt{a^n+b^n}}{(a+b)^m}+\frac{\sqrt{b^n+c^n}}{(b+c)^m}}$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{c} x=\frac{\sqrt{a^n+b^n}}{(a+b)^m}, y=\frac{\sqrt{b^n+c^n}}{(b+c)^m}, z=\frac{\sqrt{c^n+a^n}}{(c+a)^m}, \\ A=\frac{1}{(a+b)^m}, B=\frac{1}{(b+c)^m}, C=\frac{1}{(c+a)^m} \end{array}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$\text{4F. } \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{1}{(a+b)^m}.\frac{1}{(b+c)^m}\right)}$$

$$\overset{\text{Radon}}{\geq}\sqrt{3}.\sqrt{\frac{(1+1+1)^{m+1}}{\left(\sum_{cyc}\left((a+b)(b+c)\right)\right)^m}}=\sqrt{3}.\sqrt{\frac{3^{m+1}}{\left(\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab\right)^m}}$$

$$\geq\sqrt{3}.\sqrt{\frac{3^{m+1}}{\left(\frac{4}{3}\left(\sum_{cyc}a\right)^2\right)^m}} \therefore \boxed{\text{LHS} \overset{(*)}{\geq} \sqrt{3}.\sqrt{\frac{3^{m+1}}{\left(\frac{4}{3}\right)^m.\left(\left(\sum_{cyc}a\right)^m\right)^2}}}$$

$$\boxed{Case\ 1}\ m=0 \text{ and then : LHS} \overset{\text{via }(*)}{\geq} \sqrt{3}.\sqrt{\frac{3}{\left(\frac{4}{3}\right)^0.\left(\sum_{cyc}a\right)^0}}=3$$

$$=\frac{9}{2^m.(a^m+b^m+c^m)}\ (\because m=0) \therefore \text{LHS} \geq \frac{9}{2^m.(a^m+b^m+c^m)}$$

$$\boxed{Case\ 2}\ m\in\mathbb{N}^* \text{ and then : LHS} \overset{\text{via }(*)}{\geq} \sqrt{3}.\sqrt{\frac{3^{m+1}}{\left(\frac{4}{3}\right)^m.\left(\left(\sum_{cyc}a\right)^m\right)^2}} \overset{\text{Holder}}{\geq}$$

$$\sqrt{3}.\sqrt{\frac{3^{m+1}}{\left(\frac{4}{3}\right)^m.\left(3^{m-1}.\left(\sum_{cyc}a^m\right)\right)^2}}=\sqrt{3}.\sqrt{\frac{\frac{3^{2m+1}}{3^{2m-2}}}{2^{2m}.\left(\sum_{cyc}a^m\right)^2}}=\frac{9}{2^m.(a^m+b^m+c^m)}$$

$$\therefore \text{combining both cases, } \frac{\left(\frac{b+c}{a+b}\right)^m+\left(\frac{c+a}{a+b}\right)^m}{(b+c)^m.\sqrt{\frac{c^n+a^n}{a^n+b^n}}+(c+a)^m.\sqrt{\frac{b^n+c^n}{a^n+b^n}}}+$$

$$\frac{\left(\frac{c+a}{b+c}\right)^m+\left(\frac{a+b}{b+c}\right)^m}{(c+a)^m.\sqrt{\frac{a^n+b^n}{b^n+c^n}}+(a+b)^m.\sqrt{\frac{c^n+a^n}{b^n+c^n}}}+\frac{\left(\frac{a+b}{a+c}\right)^m+\left(\frac{b+c}{a+c}\right)^m}{(a+b)^m.\sqrt{\frac{b^n+c^n}{a^n+c^n}}+(b+c)^m.\sqrt{\frac{a^n+b^n}{a^n+c^n}}}$$

$$\geq\frac{9}{2^m.(a^m+b^m+c^m)}\ \forall a,b,c>0 \text{ and } m,n\in\mathbb{N}, ''='' \text{ iff } a=b=c \text{ (QED)}$$