ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a^n + b^n + c^n = 3$, then prove that :

$$\frac{1}{a^{2n-2}+b^{2n-1}+c^{2n}}+\frac{1}{b^{2n-2}+c^{2n-1}+a^{2n}}+\frac{1}{c^{2n-2}+a^{2n-1}+b^{2n}} \le 1$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, x, y, z > 0, \left(x^2 b + a^2 y^2 + z^2 a^2 b\right)\left(a^2 + b + 1\right) \overset{?}{\ge} a^2 b(x+y+z)^2$$
$$\Leftrightarrow \left(a^4 b z^2 - 2a^2 bxz + bx^2\right) + \left(a^2 b^2 z^2 - 2a^2 byz + a^2 y^2\right)$$
$$+\left(a^4 y^2 - 2a^2 bxy + b^2 x^2\right) \overset{?}{\ge} 0$$
$$\Leftrightarrow b\left(a^2 z - x\right)^2 + a^2 (bz - y)^2 + \left(a^2 y - bx\right)^2 \overset{?}{\ge} 0 \to \text{true}$$
$$\therefore \boxed{\left(x^2 b + a^2 y^2 + z^2 a^2 b\right)\left(a^2 + b + 1\right) \ge a^2 b(x+y+z)^2} \to (1)$$

$$\text{Also, } \left(x^2 b^2 c + y^2 c + z^2 b^2\right)\left(b^2 + c + 1\right) \overset{?}{\ge} b^2 c(x+y+z)^2$$
$$\Leftrightarrow \left(b^4 c x^2 - 2b^2 cxy + cy^2\right) + \left(b^2 c^2 x^2 - 2b^2 cxz + b^2 z^2\right)$$
$$+\left(b^4 z^2 - 2b^2 cyz + c^2 y^2\right) \overset{?}{\ge} 0$$
$$\Leftrightarrow c\left(b^2 x - y\right)^2 + b^2 (cx - z)^2 + \left(b^2 z - cy\right)^2 \overset{?}{\ge} 0 \to \text{true}$$
$$\therefore \boxed{\left(x^2 b^2 c + y^2 c + z^2 b^2\right)\left(b^2 + c + 1\right) \ge b^2 c(x+y+z)^2} \to (2)$$

$$\text{Again, } \left(y^2 c^2 a + z^2 a + x^2 c^2\right)\left(c^2 + a + 1\right) \overset{?}{\ge} c^2 a(x+y+z)^2$$
$$\Leftrightarrow \left(c^4 a y^2 - 2c^2 ayz + az^2\right) + \left(c^2 a^2 y^2 - 2c^2 axy + c^2 x^2\right)$$
$$+\left(c^4 x^2 - 2c^2 axz + a^2 z^2\right) \overset{?}{\ge} 0$$
$$\Leftrightarrow a\left(c^2 y - z\right)^2 + c^2 (ay - x)^2 + \left(c^2 x - az\right)^2 \overset{?}{\ge} 0 \to \text{true}$$
$$\therefore \boxed{\left(y^2 c^2 a + z^2 a + x^2 c^2\right)\left(c^2 + a + 1\right) \ge c^2 a(x+y+z)^2} \to (3)$$

Putting $x = a^n, y = b^n, z = c^n$ in (1), we arrive at : $\dfrac{a^{2n}.b + a^2.b^{2n} + c^{2n} a^2 b}{a^2 b}$

$$\ge \frac{9}{a^2 + b + 1}\ (\because x + y + z = 3) \Rightarrow \frac{1}{a^{2n-2} + b^{2n-1} + c^{2n}} \le \frac{a^2 + b + 1}{9} \to (\text{i})$$

Putting $x = a^n, y = b^n, z = c^n$ in (2), we arrive at : $\dfrac{a^{2n}.b^2 c + c.b^{2n} + c^{2n}.b^2}{b^2 c}$

$$\ge \frac{9}{b^2 + c + 1}\ (\because x + y + z = 3) \Rightarrow \frac{1}{b^{2n-2} + c^{2n-1} + a^{2n}} \le \frac{b^2 + c + 1}{9} \to (\text{ii})$$

Putting $x = a^n, y = b^n, z = c^n$ in (3), we arrive at : $\dfrac{a^{2n}.c^2 + b^{2n}.c^2 a + c^{2n}.a}{c^2 a}$

$$\ge \frac{9}{c^2 + a + 1}\ (\because x + y + z = 3) \Rightarrow \frac{1}{c^{2n-2} + a^{2n-1} + b^{2n}} \le \frac{c^2 + a + 1}{9} \to (\text{iii})$$

$$\therefore (\text{i}) + (\text{ii}) + (\text{iii}) \Rightarrow$$
$$\frac{1}{a^{2n-2}+b^{2n-1}+c^{2n}}+\frac{1}{b^{2n-2}+c^{2n-1}+a^{2n}}+\frac{1}{c^{2n-2}+a^{2n-1}+b^{2n}}$$
$$\le \sum_{\text{cyc}} \frac{a^2 + b + 1}{9} = \frac{1}{9}\left(\sum_{\text{cyc}} a^2 + \sum_{\text{cyc}} a + 3\right) \overset{?}{\le} 1 \Leftrightarrow \sum_{\text{cyc}} a^2 + \sum_{\text{cyc}} a \underset{(*)}{\overset{?}{\le}} 6$$

$$\text{Now, via Power Mean Inequality, } \left(\frac{a^3+b^3+c^3}{3}\right)^{\frac{1}{3}} \geq \left(\frac{a^2+b^2+c^2}{3}\right)^{\frac{1}{2}}$$

$$\overset{\because a^3+b^3+c^3=3}{\Rightarrow} 1 \geq \sqrt{\frac{\sum_{cyc} a^2}{3}} \Rightarrow \sum_{cyc} a^2 \leq 3 \to (\bullet)$$

$$\text{Again, via Holder, } a^3+b^3+c^3 \geq \frac{1}{9}\left(\sum_{cyc} a\right)^3 \overset{\because a^3+b^3+c^3=3}{\Rightarrow} 3 \geq \frac{1}{9}\left(\sum_{cyc} a\right)^3$$

$$\Rightarrow \sum_{cyc} a \leq 3 \to (\bullet\bullet) \therefore (\bullet)+(\bullet\bullet) \Rightarrow \text{LHS of } (*) \leq 3+3=6 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{a^{2n-2}+b^{2n-1}+c^{2n}} + \frac{1}{b^{2n-2}+c^{2n-1}+a^{2n}} + \frac{1}{c^{2n-2}+a^{2n-1}+b^{2n}} \leq 1$$

$$\forall\, a,b,c > 0 \,\middle|\, a^n+b^n+c^n = 3, \text{''}=\text{''} \text{ iff } a=b=c=1 \text{ (QED)}$$