ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ and $a\mathrm{bc} = 1$, then prove that :

$$\frac{\mathrm{c}.\sqrt{\mathrm{b}+\mathrm{c}}+\mathrm{b}.\sqrt{\mathrm{c}+a}}{\mathrm{b}(a+\mathrm{b})}+\frac{\mathrm{c}.\sqrt{a+\mathrm{b}}+a.\sqrt{\mathrm{c}+a}}{\mathrm{c}(\mathrm{b}+\mathrm{c})}+\frac{\mathrm{b}.\sqrt{a+\mathrm{b}}+a.\sqrt{\mathrm{b}+\mathrm{c}}}{a(\mathrm{c}+a)}\geq 3\sqrt{2}$$

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$\forall$ A, B, C > 0, (A + B), (B + C), (C + A) form sides of a triangle ($\because$ (A + B) + (B + C) > (C + A) and analogs) $\Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$= 6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \rightarrow (1)$$

$$\text{Now, } \forall x, y, z > 0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc}x \rightarrow \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

$$\text{We have : } \frac{\mathrm{c}.\sqrt{\mathrm{b}+\mathrm{c}}+\mathrm{b}.\sqrt{\mathrm{c}+a}}{\mathrm{b}(a+\mathrm{b})}+\frac{\mathrm{c}.\sqrt{a+\mathrm{b}}+a.\sqrt{\mathrm{c}+a}}{\mathrm{c}(\mathrm{b}+\mathrm{c})}+\frac{\mathrm{b}.\sqrt{a+\mathrm{b}}+a.\sqrt{\mathrm{b}+\mathrm{c}}}{a(\mathrm{c}+a)}$$

$$\overset{abc=1}{=} \frac{\frac{\sqrt{b+c}}{ab}+\frac{\sqrt{c+a}}{ca}}{\mathrm{b}(a+\mathrm{b})}+\frac{\frac{\sqrt{a+b}}{ab}+\frac{\sqrt{c+a}}{bc}}{\mathrm{c}(\mathrm{b}+\mathrm{c})}+\frac{\frac{\sqrt{a+b}}{ca}+\frac{\sqrt{b+c}}{bc}}{a(\mathrm{c}+a)}$$

$$\underset{\frac{1}{c}=ab}{\overset{\frac{1}{a}=bc,\ \frac{1}{b}=ca,}{=}} \frac{bc\left(\frac{\sqrt{b+c}}{b}+\frac{\sqrt{c+a}}{c}\right)}{\mathrm{b}(a+\mathrm{b})}+\frac{ca\left(\frac{\sqrt{a+b}}{a}+\frac{\sqrt{c+a}}{c}\right)}{\mathrm{c}(\mathrm{b}+\mathrm{c})}+\frac{ab\left(\frac{\sqrt{a+b}}{a}+\frac{\sqrt{b+c}}{b}\right)}{a(\mathrm{c}+a)}$$

$$= \frac{c}{a+b}.\left(\frac{\sqrt{b+c}}{b}+\frac{\sqrt{c+a}}{c}\right)+\frac{a}{b+c}.\left(\frac{\sqrt{c+a}}{c}+\frac{\sqrt{a+b}}{a}\right)$$

$$+\frac{b}{c+a}.\left(\frac{\sqrt{a+b}}{a}+\frac{\sqrt{b+c}}{b}\right)$$

$$= \frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x = c, y = a, z = b, A = \frac{\sqrt{a+b}}{a}, B = \frac{\sqrt{b+c}}{b}, C = \frac{\sqrt{c+a}}{c}\right)$$

$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB\cdot\frac{\sqrt{3}}{2}}=\sqrt{3\sum_{cyc}\left(\frac{\sqrt{a+b}}{a}\cdot\frac{\sqrt{b+c}}{b}\right)}\overset{A-G}{\geq}$$

$$3.\sqrt{\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{a^2b^2c^2}}}\overset{abc=1}{=}3.\sqrt[6]{\frac{(a+b)(b+c)(c+a)}{abc}}\overset{\text{Cesaro}}{\geq}3.\sqrt[6]{8}=3\sqrt{2}$$

$$\therefore\frac{c.\sqrt{b+c}+b.\sqrt{c+a}}{b(a+b)}+\frac{c.\sqrt{a+b}+a.\sqrt{c+a}}{c(b+c)}+\frac{b.\sqrt{a+b}+a.\sqrt{b+c}}{a(c+a)}\geq 3\sqrt{2}$$

$$\forall\, a,b,c>0 \text{ and } abc=1,''='' \text{ iff } a=b=c=1 \text{ (QED)}$$