ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ then:

$$\frac{c\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}}{b(a+b)}+\frac{c\sqrt{a^2+b^2}+a\sqrt{c^2+a^2}}{c(b+c)}+\frac{b\sqrt{a^2+b^2}+a\sqrt{b^2+c^2}}{a(c+a)}\ge 3\sqrt{2}$$

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Solution by Tapas Das-India

$$\forall x,y>0,\ \frac{(x^2+y^2)}{2}\overset{CBS}{\ge}\left(\frac{x+y}{2}\right)^2\ (1)$$

$$\frac{c\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}}{b(a+b)}\overset{(1)}{\ge}\frac{c\sqrt{\frac{(b+c)^2}{2}}+b\sqrt{\frac{(c+a)^2}{2}}}{b(a+b)}=\frac{1}{\sqrt{2}}\frac{\left(c(b+c)+b(c+a)\right)}{b(a+b)}=$$

$$=\frac{1}{\sqrt{2}}\left[\frac{c}{b}\frac{b+c}{a+b}+\frac{c+a}{a+b}\right]$$

Analogous:

$$\frac{c\sqrt{a^2+b^2}+a\sqrt{c^2+a^2}}{c(b+c)}\ge\frac{1}{\sqrt{2}}\left[\frac{a}{c}\frac{c+a}{b+c}+\frac{a+b}{b+c}\right]$$

$$\frac{b\sqrt{a^2+b^2}+a\sqrt{b^2+c^2}}{a(c+a)}\ge\frac{1}{\sqrt{2}}\left[\frac{b}{a}\frac{a+b}{c+a}+\frac{b+c}{c+a}\right]$$

By adding:

$$\frac{c\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}}{b(a+b)}+\frac{c\sqrt{a^2+b^2}+a\sqrt{c^2+a^2}}{c(b+c)}+\frac{b\sqrt{a^2+b^2}+a\sqrt{b^2+c^2}}{a(c+a)}\ge$$

$$\ge\frac{1}{\sqrt{2}}\left(\left[\frac{c}{b}\frac{b+c}{a+b}+\frac{c+a}{a+b}\right]+\left[\frac{a}{c}\frac{c+a}{b+c}+\frac{a+b}{b+c}\right]+\left[\frac{b}{a}\frac{a+b}{c+a}+\frac{b+c}{c+a}\right]\right)\overset{AM-GM}{\ge}$$

$$\ge\frac{6}{\sqrt{2}}\left(\left[\frac{c}{b}\frac{b+c}{a+b}\cdot\frac{c+a}{a+b}\right]\cdot\left[\frac{a}{c}\frac{c+a}{b+c}\cdot\frac{a+b}{b+c}\right]\cdot\left[\frac{b}{a}\frac{a+b}{c+a}\cdot\frac{b+c}{c+a}\right]\right)^{\frac{1}{6}}=$$

$$=\frac{6}{\sqrt{2}}\left(\frac{\left(abc(a+b)^2(b+c)^2(c+a)^2\right)}{abc(a+b)^2(b+c)^2(c+a)^2}\right)^{\frac{1}{6}}=3\sqrt{2}$$

$$Equality\ \ holds\ for\ a=b=c$$