ROMANIAN MATHEMATICAL MAGAZINE

$\forall\, a,b,c>0 \wedge n\in\mathbb{N}$, prove that :

$$\sum_{cyc}\frac{c.\sqrt{b^n+c^n}+b.\sqrt{c^n+a^n}}{b(a+b)}\geq 3\sqrt{2}(abc)^{\frac{n-2}{6}}$$

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$\forall\, A,B,C>0$, $(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow (1)$$

$$\text{Now, }\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{c.\sqrt{b^n+c^n}+b.\sqrt{c^n+a^n}}{b(a+b)}+\frac{c.\sqrt{a^n+b^n}+a.\sqrt{c^n+a^n}}{c(b+c)}+\frac{b.\sqrt{a^n+b^n}+a.\sqrt{b^n+c^n}}{a(c+a)}$$

$$=\frac{\frac{\sqrt{b^n+c^n}}{b}+\frac{\sqrt{c^n+a^n}}{c}}{\frac{a+b}{c}}+\frac{\frac{\sqrt{a^n+b^n}}{a}+\frac{\sqrt{c^n+a^n}}{c}}{\frac{b+c}{a}}+\frac{\frac{\sqrt{a^n+b^n}}{a}+\frac{\sqrt{b^n+c^n}}{b}}{\frac{c+a}{b}}$$

$$=\frac{c}{a+b}.\left(\frac{\sqrt{b^n+c^n}}{b}+\frac{\sqrt{c^n+a^n}}{c}\right)+\frac{a}{b+c}.\left(\frac{\sqrt{c^n+a^n}}{c}+\frac{\sqrt{a^n+b^n}}{a}\right)$$

$$+\frac{b}{c+a}.\left(\frac{\sqrt{a^n+b^n}}{a}+\frac{\sqrt{b^n+c^n}}{b}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{c} x = c, y = a, z = b, \\ A = \frac{\sqrt{a^n + b^n}}{a}, B = \frac{\sqrt{b^n + c^n}}{b}, C = \frac{\sqrt{c^n + a^n}}{c} \end{array}\right)$$

$$= \frac{x}{y+z}.\sqrt{B+C}^2 + \frac{y}{z+x}.\sqrt{C+A}^2 + \frac{z}{x+y} \overset{\text{Oppenheim}}{\geq} 4F.\sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}}$$

$$\overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB.\frac{\sqrt{3}}{2}} = \sqrt{3\sum_{cyc}\left(\frac{\sqrt{a^n+b^n}}{a}.\frac{\sqrt{b^n+c^n}}{b}\right)} \overset{A-G}{\geq}$$

$$3.\sqrt{\sqrt[3]{\frac{(a^n+b^n)(b^n+c^n)(c^n+a^n)}{a^2b^2c^2}}} \overset{\text{Cesaro}}{\geq} 3.\sqrt[6]{\frac{8(abc)^n}{(abc)^2}} = 3\sqrt{2}(abc)^{\frac{n-2}{6}}$$

$$\therefore \frac{c.\sqrt{b^n+c^n} + b.\sqrt{c^n+a^n}}{b(a+b)} + \frac{c.\sqrt{a^n+b^n} + a.\sqrt{c^n+a^n}}{c(b+c)}$$

$$+ \frac{b.\sqrt{a^n+b^n} + a.\sqrt{b^n+c^n}}{a(c+a)} \geq 3\sqrt{2}(abc)^{\frac{n-2}{6}}\ \forall\, a, b, c > 0 \text{ and } n \in \mathbb{N},$$

$$'' ='' \text{ iff } a = b = c \text{ (QED)}$$