ROMANIAN MATHEMATICAL MAGAZINE

$$If\ a,b,c>0\ and\ abc=1\ then:$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{a^5+b^5}{c^3(a^3+b^3)}+\frac{b^5+c^5}{a^3(b^3+c^3)}+\frac{c^5+a^5}{b^3(c^3+a^3)}\geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\frac{a^5+b^5}{2}\geq\frac{a^3+b^3}{2}\cdot\frac{a^2+b^2}{2}\ or$$

$$(a^5+b^5)\geq\frac{(a^3+b^3)(a^2+b^2)}{2}\ (1)$$

$$\frac{a^5+b^5}{c^3(a^3+b^3)}\overset{(1)}{\geq}\frac{(a^3+b^3)(a^2+b^2)}{2c^3(a^3+b^3)}=\frac{a^2+b^2}{2c^3}\overset{AM-GM}{\geq}\frac{2ab}{2c^3}=\frac{ab}{c^3}$$

$$Similarly,\frac{b^5+c^5}{a^3(b^3+c^3)}\geq\frac{bc}{a^3}\ and\ \frac{c^5+a^5}{b^3(c^3+a^3)}\geq\frac{ca}{b^3}$$

$$\cdot\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{a^5+b^5}{c^3(a^3+b^3)}+\frac{b^5+c^5}{a^3(b^3+c^3)}+\frac{c^5+a^5}{b^3(c^3+a^3)}\geq$$

$$\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{ab}{c^3}+\frac{bc}{a^3}+\frac{ca}{b^3}\overset{AM-GM}{\geq}6\sqrt[6]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\cdot\frac{ab}{c^3}\cdot\frac{bc}{a^3}\cdot\frac{ca}{b^3}}=$$

$$=6\sqrt[6]{\frac{1}{a^2b^2c^2}}=6(as\ \ abc=1)$$

$$Equality\ holds\ for\ a=b=c=1$$