ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{c(b+c)}{\sqrt{\frac{c}{b}}+\sqrt{\frac{b}{a}}}+\frac{a(c+a)}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{b}}}+\frac{b(a+b)}{\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{c}}}\ge 3$$

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Solution by Soumava Chakraborty-Kolkata-India

Let $a = x^2, b = y^2, c = z^2$ $(x, y, z > 0)$ and then :

$$\frac{c(b+c)}{\sqrt{\frac{c}{b}}+\sqrt{\frac{b}{a}}}+\frac{a(c+a)}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{b}}}+\frac{b(a+b)}{\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{c}}}=\frac{z^2(y^2+z^2)}{\frac{z}{y}+\frac{y}{x}}+\frac{x^2(z^2+x^2)}{\frac{x}{z}+\frac{z}{y}}+\frac{y^2(x^2+y^2)}{\frac{y}{x}+\frac{x}{z}}$$

$$\overset{xyz=1}{=}\frac{z(y^2+z^2)}{zx+y^2}+\frac{x(z^2+x^2)}{xy+z^2}+\frac{y(x^2+y^2)}{yz+x^2}\overset{CBS}{\ge}$$

$$\frac{z(y^2+z^2)}{\sqrt{y^2+z^2}.\sqrt{x^2+y^2}}+\frac{x(z^2+x^2)}{\sqrt{z^2+x^2}.\sqrt{y^2+z^2}}+\frac{y(x^2+y^2)}{\sqrt{x^2+y^2}.\sqrt{z^2+x^2}}\overset{A-G}{\ge}3\sqrt[3]{xyz}=3$$

$$\therefore \frac{c(b+c)}{\sqrt{\frac{c}{b}}+\sqrt{\frac{b}{a}}}+\frac{a(c+a)}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{b}}}+\frac{b(a+b)}{\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{c}}}\ge 3\ \forall\ a, b, c > 0 \,\middle|\, abc = 1,$$

$'' ='' $ iff $a = b = c = 1$ (QED)