ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ then:

$$\sum\frac{b^4\sqrt{b^2+c^2}+a^4\sqrt{c^2+a^2}}{a^2+b^2}\ge 3\sqrt{2}abc$$

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$$WLOG\ a\ge b\ge c\ then\ a^2+b^2\ge a^2+c^2\ge b^2+c^2$$

$$and\ \sqrt{a^2+b^2}\ge\sqrt{a^2+c^2}\ge\sqrt{b^2+c^2}$$

$$\frac{b^4\sqrt{b^2+c^2}+a^4\sqrt{c^2+a^2}}{a^2+b^2}\overset{Chebyshev}{\ge}\frac{\frac{1}{2}(b^4+a^4)\left(\sqrt{b^2+c^2}+\sqrt{a^2+c^2}\right)}{a^2+b^2}\overset{CBS}{\ge}$$

$$\ge\frac{\frac{1}{2}\frac{\left(a^2+b^2\right)^2}{2}\left(\sqrt{\frac{(b+c)^2}{2}}+\sqrt{\frac{(a+c)^2}{2}}\right)}{a^2+b^2}=$$

$$=\frac{1}{4\sqrt{2}}(a^2+b^2)(b+c+c+a)\overset{AM-GM}{\ge}\frac{1}{4\sqrt{2}}(2ab)\left(4(abc^2)^{\frac{1}{4}}\right)=\sqrt{2}\left(a^5b^5c^2\right)^{\frac{1}{4}}\ (1)$$

$$\sum\frac{b^4\sqrt{b^2+c^2}+a^4\sqrt{c^2+a^2}}{a^2+b^2}\overset{(1)}{\ge}\sqrt{2}\sum\left(a^5b^5c^2\right)^{\frac{1}{4}}\overset{AM-GM}{\ge}$$

$$\ge 3\sqrt{2}\left(a^{12}b^{12}c^{12}\right)^{\frac{1}{12}}=3\sqrt{2}abc$$

$$Equality\ holds\ for\ a=b=c$$