ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then :

$$\sum_{cyc} \frac{\left(\frac{c+a}{a+b}\right)^2 + \left(\frac{b+c}{a+b}\right)^2}{\frac{b^2}{a^2}(c+a)^2 + \frac{c^2}{a^2}(b+c)^2} \geq \frac{9}{4(a^2+b^2+c^2)}$$

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$\forall A, B, C > 0, (A+B), (B+C), (C+A)$ form sides of a triangle
$(\because (A+B)+(B+C) > (C+A)$ *and analogs*$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ form sides of a triangle with *area* F (*say*) *and* $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc} AB + B^2\right) - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB$$

$$= 6\sum_{cyc} AB + 2\sum_{cyc} A^2 - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB \Rightarrow 4F = 2\sqrt{\sum_{cyc} AB} \rightarrow (1)$$

$$\text{Now}, \forall x, y, z > 0, \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc} \frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \rightarrow \text{true} \therefore \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

$$\text{We have : } \frac{\left(\frac{c+a}{a+b}\right)^2 + \left(\frac{b+c}{a+b}\right)^2}{\frac{b^2}{a^2}(c+a)^2 + \frac{c^2}{a^2}(b+c)^2} + \frac{\left(\frac{a+b}{b+c}\right)^2 + \left(\frac{c+a}{b+c}\right)^2}{\frac{c^2}{b^2}(a+b)^2 + \frac{a^2}{b^2}(c+a)^2}$$

$$+ \frac{\left(\frac{b+c}{c+a}\right)^2 + \left(\frac{a+b}{c+a}\right)^2}{\frac{a^2}{c^2}(b+c)^2 + \frac{b^2}{c^2}(a+b)^2}$$

$$= \frac{\frac{a^2}{(a+b)^2(b+c)^2} + \frac{a^2}{(a+b)^2(c+a)^2}}{\frac{b^2}{(b+c)^2} + \frac{c^2}{(c+a)^2}} + \frac{\frac{b^2}{(b+c)^2(c+a)^2} + \frac{b^2}{(b+c)^2(a+b)^2}}{\frac{c^2}{(c+a)^2} + \frac{a^2}{(a+b)^2}} + \frac{\frac{c^2}{(c+a)^2(a+b)^2} + \frac{c^2}{(c+a)^2(b+c)^2}}{\frac{a^2}{(a+b)^2} + \frac{b^2}{(b+c)^2}}$$

$$= \frac{\frac{a^2}{(a+b)^2}\cdot\left(\frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right)}{\frac{b^2}{(b+c)^2} + \frac{c^2}{(c+a)^2}} + \frac{\frac{b^2}{(b+c)^2}\cdot\left(\frac{1}{(c+a)^2} + \frac{1}{(a+b)^2}\right)}{\frac{c^2}{(c+a)^2} + \frac{a^2}{(a+b)^2}} + \frac{\frac{c^2}{(c+a)^2}\cdot\left(\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2}\right)}{\frac{a^2}{(a+b)^2} + \frac{b^2}{(b+c)^2}}$$

$$= \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{c} x = \frac{a^2}{(a+b)^2}, y = \frac{b^2}{(b+c)^2}, z = \frac{c^2}{(c+a)^2}, \\ A = \frac{1}{(a+b)^2}, B = \frac{1}{(b+c)^2}, C = \frac{1}{(c+a)^2} \end{array}\right)$$

$$= \frac{x}{y+z}.\sqrt{B+C}^2 + \frac{y}{z+x}.\sqrt{C+A}^2 + \frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2} = \sqrt{3\sum_{cyc}\left(\frac{1}{(a+b)^2}.\frac{1}{(b+c)^2}\right)}$$

$$\overset{\text{Radon}}{\geq} \sqrt{3}.\sqrt{\frac{(1+1+1)^3}{\left(\sum_{cyc}\left((a+b)(b+c)\right)\right)^2}} = 9.\sqrt{\frac{1}{\left(\left(\sum_{cyc}a\right)^2 + \sum_{cyc}ab\right)^2}}$$

$$\geq 9.\sqrt{\frac{1}{\left(3\sum_{cyc}a^2 + \sum_{cyc}a^2\right)^2}} = \frac{9}{4(a^2+b^2+c^2)}$$

$$\therefore \frac{\left(\frac{c+a}{a+b}\right)^2 + \left(\frac{b+c}{a+b}\right)^2}{\frac{b^2}{a^2}(c+a)^2 + \frac{c^2}{a^2}(b+c)^2} + \frac{\left(\frac{a+b}{b+c}\right)^2 + \left(\frac{c+a}{b+c}\right)^2}{\frac{c^2}{b^2}(a+b)^2 + \frac{a^2}{b^2}(c+a)^2} + \frac{\left(\frac{b+c}{c+a}\right)^2 + \left(\frac{a+b}{c+a}\right)^2}{\frac{a^2}{c^2}(b+c)^2 + \frac{b^2}{c^2}(a+b)^2}$$

$$\geq \frac{9}{4(a^2+b^2+c^2)} \ \forall\, a,b,c > 0'' ='' \text{ iff } a = b = c \text{ (QED)}$$