ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $abc=1$, then :

$$\frac{a^4+a^2c^2}{a^2b^4+b^2c^4}+\frac{b^4+a^2b^2}{b^2c^4+c^2a^4}+\frac{c^4+b^2c^2}{c^2a^4+a^2b^4}\ge 3$$

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$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have : }\frac{a^4+a^2c^2}{a^2b^4+b^2c^4}+\frac{b^4+a^2b^2}{b^2c^4+c^2a^4}+\frac{c^4+b^2c^2}{c^2a^4+a^2b^4}$$

$$=\frac{x^2+xz}{xy^2+yz^2}+\frac{y^2+xy}{yz^2+zx^2}+\frac{z^2+yz}{zx^2+xy^2}\ (x=a^2,y=b^2,c=z^2)\overset{xyz=1}{=}$$

$$\frac{\frac{x}{yz}+\frac{1}{y}}{\frac{xy}{zx}+\frac{yz}{xy}}+\frac{\frac{y}{zx}+\frac{1}{z}}{\frac{yz}{xy}+\frac{zx}{yz}}+\frac{\frac{z}{xy}+\frac{1}{x}}{\frac{zx}{yz}+\frac{xy}{zx}}=\frac{\frac{1}{y}\left(\frac{x}{z}+1\right)}{\frac{y}{z}+\frac{z}{x}}+\frac{\frac{1}{z}\left(\frac{y}{x}+1\right)}{\frac{z}{x}+\frac{x}{y}}+\frac{\frac{1}{x}\left(\frac{z}{y}+1\right)}{\frac{x}{y}+\frac{y}{z}}$$

$$=\frac{\frac{x}{y}\left(\frac{1}{z}+\frac{1}{x}\right)}{\frac{y}{z}+\frac{z}{x}}+\frac{\frac{y}{z}\left(\frac{1}{x}+\frac{1}{y}\right)}{\frac{z}{x}+\frac{x}{y}}+\frac{\frac{z}{x}\left(\frac{1}{y}+\frac{1}{z}\right)}{\frac{x}{y}+\frac{y}{z}}$$

$$=\frac{X}{Y+Z}(B+C)+\frac{Y}{Z+X}(C+A)+\frac{Z}{X+Y}(A+B)$$

$$\left(X=\frac{x}{y},Y=\frac{y}{z},Z=\frac{z}{x},A=\frac{1}{y},B=\frac{1}{z},C=\frac{1}{x}\right)$$

$$= \frac{X}{Y+Z}.\sqrt{B+C}^2 + \frac{Y}{Z+X}.\sqrt{C+A}^2 + \frac{Z}{X+Y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}} \overset{via\ (1)\ and\ (2)}{\geq} 2\sqrt{\sum_{cyc} AB.\frac{\sqrt{3}}{2}} = \sqrt{3\sum_{cyc}\frac{1}{yz}}$$

$$\overset{A-G}{\geq} \sqrt{3}.\sqrt{3.\sqrt[3]{\frac{1}{x^2y^2z^2}}} \overset{xyz=1}{=} 3 \therefore \frac{a^4+a^2c^2}{a^2b^4+b^2c^4} + \frac{b^4+a^2b^2}{b^2c^4+c^2a^4} + \frac{c^4+b^2c^2}{c^2a^4+a^2b^4} \geq 3$$

$$\forall\, a,b,c > 0 \,\big|\, abc = 1, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$