ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then :

$$\frac{a^5+a^2c^3}{a^3b^4c+ab^2c^5}+\frac{b^5+b^2a^3}{ab^3c^4+a^5bc^2}+\frac{c^5+c^2b^3}{a^4bc^3+a^2b^5c}\ge\frac{3}{abc}$$

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$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{a^5+a^2c^3}{a^3b^4c+ab^2c^5}+\frac{b^5+b^2a^3}{ab^3c^4+a^5bc^2}+\frac{c^5+c^2b^3}{a^4bc^3+a^2b^5c}$$

$$=\frac{a^2(a^3+c^3)}{ab^2c(a^2b^2+c^4)}+\frac{b^2(b^3+a^3)}{abc^2(b^2c^2+a^4)}+\frac{c^2(c^3+b^3)}{a^2bc(c^2a^2+b^4)}$$

$$=\frac{\frac{a^2}{b^2}\cdot\left(\frac{a^3+c^3}{ac}\right)}{a^2b^2c^2\left(\frac{1}{c^2}+\frac{c^2}{a^2b^2}\right)}+\frac{\frac{b^2}{c^2}\cdot\left(\frac{b^3+a^3}{ab}\right)}{a^2b^2c^2\left(\frac{1}{a^2}+\frac{a^2}{b^2c^2}\right)}+\frac{\frac{c^2}{a^2}\cdot\left(\frac{c^3+b^3}{bc}\right)}{a^2b^2c^2\left(\frac{1}{b^2}+\frac{b^2}{c^2a^2}\right)}$$

$$=\frac{\frac{a^2}{b^2}\cdot\left(\frac{a^3+c^3}{a^3c^3}\right)}{\frac{b^2}{c^2}+\frac{c^2}{a^2}}+\frac{\frac{b^2}{c^2}\cdot\left(\frac{b^3+a^3}{a^3b^3}\right)}{\frac{c^2}{a^2}+\frac{a^2}{b^2}}+\frac{\frac{c^2}{a^2}\cdot\left(\frac{c^3+b^3}{b^3c^3}\right)}{\frac{a^2}{b^2}+\frac{b^2}{c^2}}$$

$$=\frac{\frac{a^2}{b^2}\cdot\left(\frac{1}{c^3}+\frac{1}{a^3}\right)}{\frac{b^2}{c^2}+\frac{c^2}{a^2}}+\frac{\frac{b^2}{c^2}\cdot\left(\frac{1}{a^3}+\frac{1}{b^3}\right)}{\frac{c^2}{a^2}+\frac{a^2}{b^2}}+\frac{\frac{c^2}{a^2}\cdot\left(\frac{1}{b^3}+\frac{1}{c^3}\right)}{\frac{a^2}{b^2}+\frac{b^2}{c^2}}$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{a^2}{b^2},y=\frac{b^2}{c^2},z=\frac{c^2}{a^2},A=\frac{1}{b^3},B=\frac{1}{c^3},C=\frac{1}{a^3}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}=\sqrt{3\sum_{cyc}\left(\frac{1}{b^3}.\frac{1}{c^3}\right)}\overset{A-G}{\geq}$$

$$3.\sqrt{\sqrt[3]{\frac{1}{a^6b^6c^6}}}\therefore\frac{a^5+a^2c^3}{a^3b^4c+ab^2c^5}+\frac{b^5+b^2a^3}{ab^3c^4+a^5bc^2}+\frac{c^5+c^2b^3}{a^4bc^3+a^2b^5c}\geq\frac{3}{abc}$$

$$\forall\, a,b,c>0, ''='' \text{ iff } a=b=c\ (\text{QED})$$