ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ and $a^5 + \mathrm{b}^5 + \mathrm{c}^5 = 3$, then prove that :

$$\sqrt[3]{a^2(a^5 + a^3\mathrm{b}^2 + \mathrm{b}^5)} + \sqrt[3]{\mathrm{b}^2(\mathrm{b}^5 + \mathrm{b}^3\mathrm{c}^2 + \mathrm{c}^5)} + \sqrt[3]{\mathrm{c}^2(\mathrm{c}^5 + \mathrm{c}^3a^2 + a^5)}$$

$$\leq 3\sqrt[3]{a^2 + \mathrm{b}^2 + \mathrm{c}^2}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via Holder}, \left(\sum_{\text{cyc}} x\right)^3 \leq 9\sum_{\text{cyc}} x^3 \ \forall\, x, \mathrm{y}, \mathrm{z} > 0 \Rightarrow \sum_{\text{cyc}} x \leq \sqrt[3]{9\sum_{\text{cyc}} x^3}$$

$$\therefore \sum_{\text{cyc}} \sqrt[3]{a^2(a^5 + a^3\mathrm{b}^2 + \mathrm{b}^5)} \leq \sqrt[3]{9\sum_{\text{cyc}} a^2(a^5 + a^3\mathrm{b}^2 + \mathrm{b}^5)} \overset{a^5 + \mathrm{b}^5 + \mathrm{c}^5 = 3}{=}$$

$$\sqrt[3]{9\sum_{\text{cyc}} a^2(3 - \mathrm{c}^5 + a^3\mathrm{b}^2)} = \sqrt[3]{27\sum_{\text{cyc}} a^2 - 9\sum_{\text{cyc}} \mathrm{c}^5 a^2 + 9\sum_{\text{cyc}} a^5\mathrm{b}^2}$$

$$= \sqrt[3]{27\sum_{\text{cyc}} a^2 - 9\sum_{\text{cyc}} a^5\mathrm{b}^2 + 9\sum_{\text{cyc}} a^5\mathrm{b}^2} = 3\sqrt[3]{a^2 + \mathrm{b}^2 + \mathrm{c}^2},$$

$'' ='' \ \text{iff } a = \mathrm{b} = \mathrm{c} = 1$ (QED)