ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$, *then* :

$$\frac{ab}{a^3+b^3}\cdot\left(\frac{b^2}{c}+\frac{a^4}{c^3}\right)+\frac{bc}{b^3+c^3}\cdot\left(\frac{c^2}{a}+\frac{b^4}{a^3}\right)+\frac{ca}{c^3+a^3}\cdot\left(\frac{a^2}{b}+\frac{c^4}{b^3}\right)\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C > 0, $(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ *and analogs*$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with *area* F (*say*) *and* $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have : }\frac{ab}{a^3+b^3}\cdot\left(\frac{b^2}{c}+\frac{a^4}{c^3}\right)+\frac{bc}{b^3+c^3}\cdot\left(\frac{c^2}{a}+\frac{b^4}{a^3}\right)+\frac{ca}{c^3+a^3}\cdot\left(\frac{a^2}{b}+\frac{c^4}{b^3}\right)$$

$$=\frac{a^3b^3}{c^3a^3+b^3c^3}\cdot\left(\frac{b^2c^2+a^4}{a^2b^2}\right)+\frac{b^3c^3}{a^3b^3+c^3a^3}\cdot\left(\frac{c^2a^2+b^4}{b^2c^2}\right)+\frac{c^3a^3}{b^3c^3+a^3b^3}\cdot\left(\frac{a^2b^2+c^4}{c^2a^2}\right)$$

$$=\frac{a^3b^3}{b^3c^3+c^3a^3}\cdot\left(\frac{c^2}{a^2}+\frac{a^2}{b^2}\right)+\frac{b^3c^3}{c^3a^3+a^3b^3}\cdot\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}\right)+\frac{c^3a^3}{a^3b^3+b^3c^3}\cdot\left(\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=a^3b^3,y=b^3c^3,z=c^3a^3,A=\frac{b^2}{c^2},B=\frac{c^2}{a^2},C=\frac{a^2}{b^2}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$\mathbf{4F.}\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{via\ (1)\ and\ (2)}{\geq}2\sqrt{\sum_{cyc}AB}\cdot\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}\right)}$$

$$\overset{A-G}{\geq}3.\sqrt{\sqrt[3]{\frac{b^2}{a^2}\cdot\frac{c^2}{b^2}\cdot\frac{a^2}{c^2}}}\ \therefore\ \frac{ab}{a^3+b^3}\cdot\left(\frac{b^2}{c}+\frac{a^4}{c^3}\right)+\frac{bc}{b^3+c^3}\cdot\left(\frac{c^2}{a}+\frac{b^4}{a^3}\right)$$

$$+\frac{ca}{c^3+a^3}\cdot\left(\frac{a^2}{b}+\frac{c^4}{b^3}\right)\geq 3\ \forall\ a,b,c>0,''=''\ \text{iff}\ \ a=b=c\ (\text{QED})$$