ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\frac{a^{n+1}b^nc + c^{2n+1}a}{b^{n+1}(a^{n+1}+c^{n+1})} + \frac{b^{n+1}c^na + a^{2n+1}b}{c^{n+1}(a^{n+1}+b^{n+1})} + \frac{c^{n+1}a^nb + b^{2n+1}\,c}{a^{n+1}(b^{n+1}+c^{n+1})} \ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

Walter Janous inequality: a, b, c *and* x, y, z *be positive real numbers then*

$$\frac{x}{y+z}(b+c) + \frac{y}{z+x}(c+a) + \frac{z}{x+y}(a+b) \ge \sqrt{3(ab+bc+ca)}\ (1)$$

$$\frac{a^{n+1}b^nc + c^{2n+1}a}{b^{n+1}(a^{n+1}+c^{n+1})} = \frac{1}{b^{n+1}}\frac{\frac{a^{n+1}b^nc+c^{2n+1}a}{a^{n+1}c^{n+1}}}{\frac{(a^{n+1}+c^{n+1})}{a^{n+1}c^{n+1}}} = \frac{1}{b^{n+1}}\frac{\left(\frac{b^n}{c^n}+\frac{c^n}{a^n}\right)}{\frac{1}{a^{n+1}}+\frac{1}{c^{n+1}}} =$$

$$= \frac{\frac{1}{b^{n+1}}}{\frac{1}{a^{n+1}}+\frac{1}{c^{n+1}}}\left(\frac{b^n}{c^n}+\frac{c^n}{a^n}\right)\ (2)$$

Similarly $\displaystyle\frac{b^{n+1}c^na + a^{2n+1}b}{c^{n+1}(a^{n+1}+b^{n+1})} = \frac{\frac{1}{c^{n+1}}}{\frac{1}{a^{n+1}}+\frac{1}{b^{n+1}}}\left(\frac{c^n}{a^n}+\frac{a^n}{b^n}\right)$ *and*

$$\frac{c^{n+1}a^nb + b^{2n+1}\,c}{a^{n+1}(b^{n+1}+c^{n+1})} = \frac{\frac{1}{a^{n+1}}}{\frac{1}{b^{n+1}}+\frac{1}{c^{n+1}}}\left(\frac{b^n}{c^n}+\frac{a^n}{b^n}\right)$$

$$\frac{a^{n+1}b^nc + c^{2n+1}a}{b^{n+1}(a^{n+1}+c^{n+1})} + \frac{b^{n+1}c^na + a^{2n+1}b}{c^{n+1}(a^{n+1}+b^{n+1})} + \frac{c^{n+1}a^nb + b^{2n+1}\,c}{a^{n+1}(b^{n+1}+c^{n+1})} =$$

$$= \frac{\frac{1}{b^{n+1}}}{\frac{1}{a^{n+1}}+\frac{1}{c^{n+1}}}\left(\frac{b^n}{c^n}+\frac{c^n}{a^n}\right) + \frac{\frac{1}{c^{n+1}}}{\frac{1}{a^{n+1}}+\frac{1}{b^{n+1}}}\left(\frac{c^n}{a^n}+\frac{a^n}{b^n}\right) + \frac{\frac{1}{a^{n+1}}}{\frac{1}{b^{n+1}}+\frac{1}{c^{n+1}}}\left(\frac{b^n}{c^n}+\frac{a^n}{b^n}\right) \overset{(1)}{\ge}$$

$$\ge \sqrt{3\left(\left(\frac{b^n}{c^n}\cdot\frac{c^n}{a^n}\right)+\left(\frac{c^n}{a^n}\cdot\frac{a^n}{b^n}\right)+\left(\frac{b^n}{c^n}\cdot\frac{a^n}{b^n}\right)\right)} \overset{AM-GM}{\ge}$$

$$\ge \sqrt{9\left(\left(\left(\frac{b^n}{c^n}\cdot\frac{c^n}{a^n}\right)\cdot\left(\frac{c^n}{a^n}\cdot\frac{a^n}{b^n}\right)\cdot\left(\frac{b^n}{c^n}\cdot\frac{a^n}{b^n}\right)\right)^{\frac{1}{3}}\right)} = 3$$

Equality holds for $a = b = c$