ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $\forall\ n\in\mathbb{N}$, then prove that :

$$\frac{a^{2n+1}+a^nc^{n+1}}{b^{2n}a^{n+1}c+c^{2n+1}b^na}+\frac{b^{2n+1}+b^na^{n+1}}{c^{2n}b^{n+1}a+a^{2n+1}c^nb}+\frac{c^{2n+1}+c^nb^{n+1}}{a^{2n}c^{n+1}b+b^{2n+1}a^nc}\geq\frac{3}{(abc)^{\frac{n+1}{3}}}$$

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$$\forall\ A',B',C'>0,(A'+B'),(B'+C'),(C'+A')\text{ form sides of a triangle}$$
$$(\because (A'+B')+(B'+C')>(C'+A')\text{ and analogs})$$
$$\Rightarrow \sqrt{A'+B'},\sqrt{B'+C'},\sqrt{C'+A'}\text{ form sides of a triangle with area F (say) and }16F^2$$
$$=2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$
$$=2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right)-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'$$
$$=6\sum_{cyc}A'B'+2\sum_{cyc}A'^2-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'\Rightarrow 4F=2\sqrt{\sum_{cyc}A'B'}\rightarrow(1)$$

$$\text{Now, }\forall\ x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{a^{2n+1}+a^nc^{n+1}}{b^{2n}a^{n+1}c+c^{2n+1}b^na}+\frac{b^{2n+1}+b^na^{n+1}}{c^{2n}b^{n+1}a+a^{2n+1}c^nb}+\frac{c^{2n+1}+c^nb^{n+1}}{a^{2n}c^{n+1}b+b^{2n+1}a^nc}$$
$$=\frac{a^n(a^{n+1}+c^{n+1})}{ab^nc(a^nb^n+c^{2n})}+\frac{b^n(b^{n+1}+a^{n+1})}{bc^na(c^nb^n+a^{2n})}+\frac{c^n(c^{n+1}+b^{n+1})}{ca^nb(a^nc^n+b^{2n})}$$
$$=\frac{a^n\left(\frac{a^{n+1}+c^{n+1}}{a^nc^n}\right)}{ab^nc\left(\frac{b^n}{c^n}+\frac{c^n}{a^n}\right)}+\frac{b^n\left(\frac{b^{n+1}+a^{n+1}}{a^nb^n}\right)}{bc^na\left(\frac{c^n}{a^n}+\frac{a^n}{b^n}\right)}+\frac{c^n\left(\frac{c^{n+1}+b^{n+1}}{b^nc^n}\right)}{ca^nb\left(\frac{a^n}{b^n}+\frac{b^n}{c^n}\right)}$$
$$=\frac{\frac{a^n}{b^n}}{\frac{b^n}{c^n}+\frac{c^n}{a^n}}\cdot\left(\frac{1}{c^{n+1}}+\frac{1}{a^{n+1}}\right)+\frac{\frac{b^n}{c^n}}{\frac{c^n}{a^n}+\frac{a^n}{b^n}}\cdot\left(\frac{1}{a^{n+1}}+\frac{1}{b^{n+1}}\right)+\frac{\frac{c^n}{a^n}}{\frac{a^n}{b^n}+\frac{b^n}{c^n}}\cdot\left(\frac{1}{b^{n+1}}+\frac{1}{c^{n+1}}\right)$$
$$=\frac{x}{y+z}(B'+C')+\frac{y}{z+x}(C'+A')+\frac{z}{x+y}(A'+B')$$
$$\left(x=\frac{a^n}{b^n},y=\frac{b^n}{c^n},z=\frac{c^n}{a^n},A'=\frac{1}{b^{n+1}},B'=\frac{1}{c^{n+1}},C'=\frac{1}{a^{n+1}}\right)$$
$$=\frac{x}{y+z}\cdot\sqrt{B'+C'}^2+\frac{y}{z+x}\cdot\sqrt{C'+A'}^2+\frac{z}{x+y}\cdot\sqrt{A'+B'}^2$$

$$\overset{\text{Oppenheim}}{\geq} 4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}A'B'.\frac{\sqrt{3}}{2}}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{1}{b^{n+1}}.\frac{1}{c^{n+1}}\right)} \overset{A-G}{\geq} 3.\sqrt[6]{\frac{1}{(a^2b^2c^2)^{n+1}}}=\frac{3}{(abc)^{\frac{n+1}{3}}}\therefore$$

$$\frac{a^{2n+1}+a^nc^{n+1}}{b^{2n}a^{n+1}c+c^{2n+1}b^na}+\frac{b^{2n+1}+b^na^{n+1}}{c^{2n}b^{n+1}a+a^{2n+1}c^nb}+\frac{c^{2n+1}+c^nb^{n+1}}{a^{2n}c^{n+1}b+b^{2n+1}a^nc}\geq\frac{3}{(abc)^{\frac{n+1}{3}}}$$

$$\forall\, a,b,c>0 \text{ and } \forall\, n\in\mathbb{N},'' ='' \text{ iff } a=b=c \text{ (QED)}$$