ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{a+b}{b^2c^2} \cdot \frac{c^2\sqrt{b+c} + b^2\sqrt{c+a}}{a+b+2c} \geq 3 \cdot \sqrt{\frac{2}{abc}}$$

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$$\forall A', B', C' > 0, (A'+B'), (B'+C'), (C'+A') \text{ form sides of } a \text{ triangle}$$
$$(\because (A'+B') + (B'+C') > (C'+A') \text{ and analogs})$$

$$\Rightarrow \sqrt{A'+B'}, \sqrt{B'+C'}, \sqrt{C'+A'} \text{ form sides of } a \text{ triangle with area F (say) and}$$

$$16F^2 = 2\sum_{cyc}(A'+B')(B'+C') - \sum_{cyc}(A'+B')^2$$
$$= 2\sum_{cyc}\left(\sum_{cyc} A'B' + B'^2\right) - 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'B'$$
$$= 6\sum_{cyc} A'B' + 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'B' \Rightarrow 4F = 2\sqrt{\sum_{cyc} A'B'} \to (1)$$

$$\text{Now}, \forall x, y, z > 0, \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc} \frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

We have :

$$\frac{a+b}{b^2c^2} \cdot \frac{c^2.\sqrt{b+c} + b^2.\sqrt{c+a}}{a+b+2c} + \frac{b+c}{c^2a^2} \cdot \frac{c^2.\sqrt{a+b} + a^2.\sqrt{c+a}}{b+c+2a}$$
$$+ \frac{c+a}{a^2b^2} \cdot \frac{b^2.\sqrt{a+b} + a^2.\sqrt{b+c}}{c+a+2b} =$$
$$= \frac{a+b}{(b+c)+(c+a)} \cdot \left(\frac{\sqrt{b+c}}{b^2} + \frac{\sqrt{c+a}}{c^2}\right) + \frac{b+c}{(c+a)+(a+b)} \cdot \left(\frac{\sqrt{c+a}}{c^2} + \frac{\sqrt{a+b}}{a^2}\right)$$
$$+ \frac{c+a}{(a+b)+(b+c)} \cdot \left(\frac{\sqrt{a+b}}{a^2} + \frac{\sqrt{b+c}}{b^2}\right)$$
$$= \frac{x}{y+z}(B'+C') + \frac{y}{z+x}(C'+A') + \frac{z}{x+y}(A'+B')$$

$$\left(x = a + b, y = b + c, z = c + a, A' = \frac{\sqrt{a+b}}{a^2}, B' = \frac{\sqrt{b+c}}{b^2}, C' = \frac{\sqrt{c+a}}{c^2}\right)$$

$$\overset{\text{Oppenheim}}{\geq} 4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} A'B'}.\frac{\sqrt{3}}{2}$$

$$= \sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{\sqrt{a+b}}{a^2}.\frac{\sqrt{b+c}}{b^2}\right)} \overset{A-G}{\geq} 3.\sqrt[6]{\frac{(a+b)(b+c)(c+a)}{a^4b^4c^4}} \overset{\text{Cesaro}}{\geq} 3.\sqrt[6]{\frac{8abc}{a^4b^4c^4}}$$

$$= 3.\sqrt{\frac{2}{abc}} \;\forall a,b,c > 0, '' = '' \text{ iff } a = b = c \text{ (QED)}$$