ROMANIAN MATHEMATICAL MAGAZINE

**If $a,\mathbf{b},\mathbf{c} > 0$, *then prove that*:**

$$\sum_{cyc}\frac{ab^2c^2+2b^2c^3+b^3c^2}{a^2c^2\sqrt{\frac{b+c}{a+b}}+a^2b^2\sqrt{\frac{c+a}{a+b}}}\ge 6\sqrt[3]{abc}$$

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$$\forall\, A',B',C'>0,\ (A'+B'),(B'+C'),(C'+A')\ \text{form sides of a triangle}$$
$$(\because\ (A'+B')+(B'+C')>(C'+A')\ \text{and analogs}) \Rightarrow \sqrt{A'+B'},\sqrt{B'+C'},\sqrt{C'+A'}$$
$$\text{form sides of a }\Delta\text{ with area F (say) and } 16F^2 =$$
$$2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$
$$=2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right)-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'$$
$$=6\sum_{cyc}A'B'+2\sum_{cyc}A'^2-2\sum_{cyc}A'^2-2\sum_{cyc}A'B' \Rightarrow 4F=2\sqrt{\sum_{cyc}A'B'}\rightarrow(1)$$
$$\text{Now, }\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$
$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=$$
$$\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\rightarrow\text{true}$$
$$\therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\rightarrow(2)$$
$$\text{We have}: \frac{ab^2c^2+2b^2c^3+b^3c^2}{a^2c^2.\sqrt{\frac{b+c}{a+b}}+a^2b^2.\sqrt{\frac{c+a}{a+b}}}+\frac{bc^2a^2+2c^2a^3+c^3a^2}{b^2c^2.\sqrt{\frac{a+b}{b+c}}+a^2b^2.\sqrt{\frac{c+a}{b+c}}}$$
$$+\frac{cb^2a^2+2a^2b^3+a^3b^2}{b^2c^2.\sqrt{\frac{a+b}{a+c}}+a^2c^2.\sqrt{\frac{b+c}{a+c}}}$$
$$=\frac{a+2c+b}{\frac{a^2}{b^2}.\sqrt{\frac{b+c}{a+b}}+\frac{a^2}{c^2}.\sqrt{\frac{c+a}{a+b}}}+\frac{b+2a+c}{\frac{b^2}{a^2}.\sqrt{\frac{a+b}{b+c}}+\frac{b^2}{c^2}.\sqrt{\frac{c+a}{b+c}}}+\frac{c+2b+a}{\frac{c^2}{a^2}.\sqrt{\frac{a+b}{a+c}}+\frac{c^2}{b^2}.\sqrt{\frac{b+c}{a+c}}}$$

$$=\frac{\frac{\sqrt{a+b}}{a^2}.(b+c+c+a)}{\frac{\sqrt{b+c}}{b^2}+\frac{\sqrt{c+a}}{c^2}}+\frac{\frac{\sqrt{b+c}}{b^2}.(c+a+a+b)}{\frac{\sqrt{c+a}}{c^2}+\frac{\sqrt{a+b}}{a^2}}+\frac{\frac{\sqrt{c+a}}{c^2}.(a+b+b+c)}{\frac{\sqrt{a+b}}{a^2}+\frac{\sqrt{b+c}}{b^2}}$$

$$=\frac{x}{y+z}(B'+C')+\frac{y}{z+x}(C'+A')+\frac{z}{x+y}(A'+B')$$

$$\left(x=\frac{\sqrt{a+b}}{a^2},y=\frac{\sqrt{b+c}}{b^2},z=\frac{\sqrt{c+a}}{c^2},A'=a+b,B'=b+c,C'=c+a\right)$$

$$\overset{\text{Oppenheim}}{\geq} 4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}A'B'}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}\big((a+b)(b+c)\big)}\overset{A-G}{\geq} 3.\sqrt[6]{(a+b)^2(b+c)^2(c+a)^2}\overset{\text{Cesaro}}{\geq} 3.\sqrt[6]{64a^2b^2c^2}$$

$$=6.\sqrt[3]{abc}\ \forall\ a,b,c>0,''=''\ \text{iff}\ a=b=c\ (\text{QED})$$