ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ and $a\mathrm{bc} = 1$, then prove that :

$$\frac{\sqrt{b}.\sqrt[3]{\frac{a}{b}}+\sqrt{c}.\sqrt[3]{\frac{c}{b}}}{\sqrt[3]{c}.\sqrt[5]{\frac{c}{a}}+\sqrt[3]{a}.\sqrt[5]{\frac{b}{a}}}+\frac{\sqrt{c}.\sqrt[3]{\frac{b}{c}}+\sqrt{a}.\sqrt[3]{\frac{a}{c}}}{\sqrt[3]{a}.\sqrt[5]{\frac{a}{b}}+\sqrt[3]{b}.\sqrt[5]{\frac{c}{b}}}+\frac{\sqrt{a}.\sqrt[3]{\frac{c}{a}}+\sqrt{b}.\sqrt[3]{\frac{b}{a}}}{\sqrt[3]{b}.\sqrt[5]{\frac{b}{c}}+\sqrt[3]{c}.\sqrt[5]{\frac{a}{c}}}\geq 3$$

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$\forall$ A', B', C' > 0, (A' + B'), (B' + C'), (C' + A') form sides of a triangle
$(\because (A'+B')+(B'+C') > (C'+A')$ and analogs$) \Rightarrow \sqrt{A'+B'}, \sqrt{B'+C'}, \sqrt{C'+A'}$
form sides of a Δ with area F (say) and

$$16F^2 = 2\sum_{cyc}(A'+B')(B'+C') - \sum_{cyc}(A'+B')^2$$

$$= 2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right) - 2\sum_{cyc}A'^2 - 2\sum_{cyc}A'B'$$

$$= 6\sum_{cyc}A'B' + 2\sum_{cyc}A'^2 - 2\sum_{cyc}A'^2 - 2\sum_{cyc}A'B' \Rightarrow 4F = 2\sqrt{\sum_{cyc}A'B'} \rightarrow (1)$$

Now, $\forall$ X, Y, Z > 0, $\sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{X^2Y^2}{XY(Y+Z)(Z+X)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$

Via Bergstrom, LHS of $(*) \geq \frac{\left(\sum_{cyc}XY\right)^2}{\sum_{cyc}\left(XY\left(\sum_{cyc}XY+Z^2\right)\right)} = \frac{\left(\sum_{cyc}XY\right)^2}{\left(\sum_{cyc}XY\right)^2 + XYZ\sum_{cyc}X}$

$$\overset{?}{\geq}\frac{3}{4} \Leftrightarrow \left(\sum_{cyc}XY\right)^2 \overset{?}{\geq} 3XYZ\sum_{cyc}X \rightarrow \text{true} \therefore \sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}} \geq \frac{\sqrt{3}}{2} \rightarrow (2)$$

We have : $\frac{\sqrt{b}.\sqrt[3]{\frac{a}{b}}+\sqrt{c}.\sqrt[3]{\frac{c}{b}}}{\sqrt[3]{c}.\sqrt[5]{\frac{c}{a}}+\sqrt[3]{a}.\sqrt[5]{\frac{b}{a}}}+\frac{\sqrt{c}.\sqrt[3]{\frac{b}{c}}+\sqrt{a}.\sqrt[3]{\frac{a}{c}}}{\sqrt[3]{a}.\sqrt[5]{\frac{a}{b}}+\sqrt[3]{b}.\sqrt[5]{\frac{c}{b}}}+\frac{\sqrt{a}.\sqrt[3]{\frac{c}{a}}+\sqrt{b}.\sqrt[3]{\frac{b}{a}}}{\sqrt[3]{b}.\sqrt[5]{\frac{b}{c}}+\sqrt[3]{c}.\sqrt[5]{\frac{a}{c}}}$

$$= \frac{y^{15}\left(\frac{x}{y}\right)^{10}+z^{15}\left(\frac{z}{y}\right)^{10}}{z^{10}\left(\frac{z}{x}\right)^{6}+x^{10}\left(\frac{y}{x}\right)^{6}}+\frac{z^{15}\left(\frac{y}{z}\right)^{10}+x^{15}\left(\frac{x}{z}\right)^{10}}{x^{10}\left(\frac{x}{y}\right)^{6}+y^{10}\left(\frac{z}{y}\right)^{6}}+\frac{x^{15}\left(\frac{z}{x}\right)^{10}+y^{15}\left(\frac{y}{x}\right)^{10}}{y^{10}\left(\frac{y}{z}\right)^{6}+z^{10}\left(\frac{x}{z}\right)^{6}}$$

$$\left(x = \sqrt[30]{a}, y = \sqrt[30]{b}, z = \sqrt[30]{c}\right)$$

$$= \frac{x^{16}y^{15}+x^6z^{25}}{z^{16}y^{10}+x^{10}y^{16}}+\frac{y^{16}z^{15}+y^6x^{25}}{x^{16}z^{10}+y^{10}z^{16}}+\frac{z^{16}x^{15}+z^6y^{25}}{y^{16}x^{10}+z^{10}x^{16}}$$

$$= \frac{x^{16}y^{15}+x^6z^{25}}{z^6(yz)^{10}+y^6(xy)^{10}}+\frac{y^{16}z^{15}+y^6x^{25}}{x^6(zx)^{10}+z^6(yz)^{10}}+\frac{z^{16}x^{15}+z^6y^{25}}{y^6(xy)^{10}+x^6(zx)^{10}}$$

$$\overset{xyz=1}{=} \frac{x^{16}y^{15}+x^6z^{25}}{\frac{z^6}{x^{10}}+\frac{y^6}{z^{10}}}+\frac{y^{16}z^{15}+y^6x^{25}}{\frac{x^6}{y^{10}}+\frac{z^6}{x^{10}}}+\frac{z^{16}x^{15}+z^6y^{25}}{\frac{y^6}{z^{10}}+\frac{x^6}{y^{10}}}$$

$$=\frac{\frac{x^6}{y^{10}}\left(x^{10}y^{25}+y^{10}z^{25}\right)}{\frac{y^6}{z^{10}}+\frac{z^6}{x^{10}}}+\frac{\frac{y^6}{z^{10}}\left(y^{10}z^{25}+z^{10}x^{25}\right)}{\frac{z^6}{x^{10}}+\frac{x^6}{y^{10}}}+\frac{\frac{z^6}{x^{10}}\left(z^{10}x^{25}+x^{10}y^{25}\right)}{\frac{x^6}{y^{10}}+\frac{y^6}{z^{10}}}$$

$$=\frac{X}{Y+Z}(B'+C')+\frac{Y}{Z+X}(C'+A')+\frac{Z}{X+Y}(A'+B')$$

$$\left(X=\frac{x^6}{y^{10}},Y=\frac{y^6}{z^{10}},Z=\frac{z^6}{x^{10}},A'=z^{10}x^{25},B'=x^{10}y^{25},C'=y^{10}z^{25}\right)\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\overset{\text{via }(1)\text{ and }(2)}{\geq}2\sqrt{\sum_{cyc}A'B'.\frac{\sqrt{3}}{2}}=\sqrt{3}.\sqrt{\sum_{cyc}(z^{10}x^{25}.x^{10}y^{25})}$$

$$\overset{xyz=1}{=}\sqrt{3}.\sqrt{\sum_{cyc}x^{25}y^{15}}\overset{A-G}{\geq}3.\sqrt[6]{(xyz)^{25}.(xyz)^{15}}\overset{xyz=1}{=}3$$

$$\therefore\frac{\sqrt{b}.\sqrt[3]{\frac{a}{b}}+\sqrt{c}.\sqrt[3]{\frac{c}{b}}}{\sqrt[3]{c}.\sqrt[5]{\frac{c}{a}}+\sqrt[3]{a}.\sqrt[5]{\frac{b}{a}}}+\frac{\sqrt{c}.\sqrt[3]{\frac{b}{c}}+\sqrt{a}.\sqrt[3]{\frac{a}{c}}}{\sqrt[3]{a}.\sqrt[5]{\frac{a}{b}}+\sqrt[3]{b}.\sqrt[5]{\frac{c}{b}}}+\frac{\sqrt{a}.\sqrt[3]{\frac{c}{a}}+\sqrt{b}.\sqrt[3]{\frac{b}{a}}}{\sqrt[3]{b}.\sqrt[5]{\frac{b}{c}}+\sqrt[3]{c}.\sqrt[5]{\frac{a}{c}}}\geq 3$$

$$\forall\, a,b,c>0\,\middle|\,abc=1,''=''\text{ iff } a=b=c=1\text{ (QED)}$$