ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then prove that :

$$\sum_{cyc}\frac{(b^3+c^3)^3.\sqrt[3]{\frac{c^3+a^3}{a^3+b^3}}+(c^3+a^3)^3.\sqrt[3]{\frac{b^3+c^3}{a^3+b^3}}}{\sqrt[3]{b^3+c^3}+\sqrt[3]{c^3+a^3}}\geq\frac{12.\sqrt[6]{48}.a^3b^3c^3}{\sqrt[6]{a^6+b^6+c^6}}$$

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$\forall A',B',C'>0,(A'+B'),(B'+C'),(C'+A')$ form sides of a triangle $(\because (A'+B')+(B'+C')>(C'+A')$ and analogs$)\Rightarrow\sqrt{A'+B'},\sqrt{B'+C'},\sqrt{C'+A'}$ form sides of a Δ with area F (say) and $16F^2=$

$$2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$

$$=2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right)-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'$$

$$=6\sum_{cyc}A'B'+2\sum_{cyc}A'^2-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'\Rightarrow 4F=2\sqrt{\sum_{cyc}A'B'}\rightarrow(1)$$

Now, $\forall X,Y,Z>0,\ \sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{X^2Y^2}{XY(Y+Z)(Z+X)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$

Via Bergstrom, LHS of $(*)\geq\frac{\left(\sum_{cyc}XY\right)^2}{\sum_{cyc}\left(XY\left(\sum_{cyc}XY+Z^2\right)\right)}=\frac{\left(\sum_{cyc}XY\right)^2}{\left(\sum_{cyc}XY\right)^2+XYZ\sum_{cyc}X}$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}XY\right)^2\overset{?}{\geq}3XYZ\sum_{cyc}X\rightarrow\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\geq\frac{\sqrt{3}}{2}\rightarrow(2)$$

We have : $\frac{(b^3+c^3)^3.\sqrt[3]{\frac{c^3+a^3}{a^3+b^3}}+(c^3+a^3)^3.\sqrt[3]{\frac{b^3+c^3}{a^3+b^3}}}{\sqrt[3]{b^3+c^3}+\sqrt[3]{c^3+a^3}}+$

$$\frac{(a^3+b^3)^3.\sqrt[3]{\frac{c^3+a^3}{b^3+c^3}}+(c^3+a^3)^3.\sqrt[3]{\frac{a^3+b^3}{b^3+c^3}}}{\sqrt[3]{a^3+b^3}+\sqrt[3]{c^3+a^3}}+\frac{(a^3+b^3)^3.\sqrt[3]{\frac{b^3+c^3}{c^3+a^3}}+(b^3+c^3)^3.\sqrt[3]{\frac{a^3+b^3}{c^3+a^3}}}{\sqrt[3]{a^3+b^3}+\sqrt[3]{b^3+c^3}}$$

$$=\frac{y^9\left(\frac{z}{x}\right)+z^9\left(\frac{y}{x}\right)}{y+z}+\frac{x^9\left(\frac{z}{y}\right)+z^9\left(\frac{x}{y}\right)}{z+x}+\frac{x^9\left(\frac{y}{z}\right)+y^9\left(\frac{x}{z}\right)}{x+y}$$

$$\left(x=\sqrt[3]{a^3+b^3},y=\sqrt[3]{b^3+c^3},z=\sqrt[3]{c^3+a^3}\right)$$

$$=\frac{y^9\left(\frac{1}{xy}\right)+z^9\left(\frac{1}{zx}\right)}{\frac{1}{z}+\frac{1}{y}}+\frac{x^9\left(\frac{1}{xy}\right)+z^9\left(\frac{1}{yz}\right)}{\frac{1}{x}+\frac{1}{z}}+\frac{x^9\left(\frac{1}{zx}\right)+y^9\left(\frac{1}{yz}\right)}{\frac{1}{y}+\frac{1}{x}}$$

$$=\frac{\frac{1}{x}.(y^8+z^8)}{\frac{1}{y}+\frac{1}{z}}+\frac{\frac{1}{y}.(z^8+x^8)}{\frac{1}{z}+\frac{1}{x}}+\frac{\frac{1}{z}.(x^8+y^8)}{\frac{1}{x}+\frac{1}{y}}$$

$$=\frac{X}{Y+Z}(B'+C')+\frac{Y}{Z+X}(C'+A')+\frac{Z}{X+Y}(A'+B')$$

$$\left(X=\frac{1}{x},Y=\frac{1}{y},Z=\frac{1}{z},A'=x^8,B'=y^8,C'=z^8\right)\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}A'B'}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}(a^3+b^3)^{\frac{8}{3}}(b^3+c^3)^{\frac{8}{3}}}\overset{\text{AM-GM}}{\geq}3.\sqrt[6]{(a^3+b^3)^{\frac{16}{3}}(b^3+c^3)^{\frac{16}{3}}(c^3+a^3)^{\frac{16}{3}}}\overset{\text{Cesaro}}{\geq}$$

$$3.\sqrt[6]{(8a^3b^3c^3)^{\frac{16}{3}}}\overset{?}{\geq}\frac{12.\sqrt[6]{48}.a^3b^3c^3}{\sqrt[6]{a^6+b^6+c^6}}\Leftrightarrow 2^{16}(abc)^{16}\left(\sum_{cyc}a^6\right)\overset{?}{\geq}2^{12}.2^4.3(abc)^{18}$$

$$\Leftrightarrow\sum_{cyc}a^6\overset{?}{\geq}3a^2b^2c^2\to\text{true via AM}-\text{GM}\therefore$$

$$\frac{(b^3+c^3)^3.\sqrt[3]{\frac{c^3+a^3}{a^3+b^3}}+(c^3+a^3)^3.\sqrt[3]{\frac{b^3+c^3}{a^3+b^3}}}{\sqrt[3]{b^3+c^3}+\sqrt[3]{c^3+a^3}}+\frac{(a^3+b^3)^3.\sqrt[3]{\frac{c^3+a^3}{b^3+c^3}}+(c^3+a^3)^3.\sqrt[3]{\frac{a^3+b^3}{b^3+c^3}}}{\sqrt[3]{a^3+b^3}+\sqrt[3]{c^3+a^3}}$$

$$+\frac{(a^3+b^3)^3.\sqrt[3]{\frac{b^3+c^3}{c^3+a^3}}+(b^3+c^3)^3.\sqrt[3]{\frac{a^3+b^3}{c^3+a^3}}}{\sqrt[3]{a^3+b^3}+\sqrt[3]{b^3+c^3}}\geq\frac{12.\sqrt[6]{48}.a^3b^3c^3}{\sqrt[6]{a^6+b^6+c^6}}\ \forall\, a,b,c>0,$$

$$''='' \text{ iff } a=b=c\ (\text{QED})$$