ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then prove that:

$$\sum_{cyc}\frac{\sqrt{\frac{c^2(b^3+c^3)}{b}}+\sqrt{\frac{b^2(c^3+a^3)}{c}}}{ac\cdot\frac{b+c}{a+b}+ab\cdot\frac{c+a}{a+b}}\ge 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall A',B',C'>0,(A'+B'),(B'+C'),(C'+A')\text{ form sides of a triangle}$$
$$(\because (A'+B')+(B'+C')>(C'+A')\text{ and analogs})\Rightarrow\sqrt{A'+B'},\sqrt{B'+C'},\sqrt{C'+A'}$$
$$\text{form sides of a }\Delta\text{ with area F (say) and }16F^2=$$
$$2\sum_{cyc}(A'+B')(B'+C')-\sum_{cyc}(A'+B')^2$$
$$=2\sum_{cyc}\left(\sum_{cyc}A'B'+B'^2\right)-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'$$
$$=6\sum_{cyc}A'B'+2\sum_{cyc}A'^2-2\sum_{cyc}A'^2-2\sum_{cyc}A'B'\Rightarrow 4F=2\sqrt{\sum_{cyc}A'B'}\to(1)$$

$$\text{Now},\forall x,y,z>0,\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=$$

$$\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}\overset{?}{\ge}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}$$

$$\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have}:\frac{\sqrt{\frac{c^2(b^3+c^3)}{b}}+\sqrt{\frac{b^2(c^3+a^3)}{c}}}{ac.\frac{b+c}{a+b}+ab.\frac{c+a}{a+b}}+\frac{\sqrt{\frac{c^2(a^3+b^3)}{a}}+\sqrt{\frac{a^2(c^3+a^3)}{c}}}{bc.\frac{a+b}{b+c}+ab.\frac{c+a}{b+c}}$$
$$+\frac{\sqrt{\frac{b^2(a^3+b^3)}{a}}+\sqrt{\frac{a^2(b^3+c^3)}{b}}}{bc.\frac{a+b}{a+c}+ac.\frac{b+c}{a+c}}$$

$$=\frac{\frac{a+b}{ab}.\sqrt{\frac{b^3+c^3}{b}}+\frac{a+b}{ca}.\sqrt{\frac{c^3+a^3}{c}}}{\frac{b+c}{b}+\frac{c+a}{c}}+\frac{\frac{b+c}{ab}.\sqrt{\frac{a^3+b^3}{a}}+\frac{b+c}{bc}.\sqrt{\frac{c^3+a^3}{c}}}{\frac{a+b}{a}+\frac{c+a}{c}}$$

$$+\frac{\frac{c+a}{ca}\cdot\sqrt{\frac{a^3+b^3}{a}}+\frac{c+a}{bc}\cdot\sqrt{\frac{b^3+c^3}{b}}}{\frac{a+b}{a}+\frac{b+c}{b}}$$

$$=\frac{\frac{a+b}{a}}{\frac{b+c}{b}+\frac{c+a}{c}}\cdot\left(\sqrt{\frac{b^3+c^3}{b^3}}+\sqrt{\frac{c^3+a^3}{c^3}}\right)+\frac{\frac{b+c}{b}}{\frac{c+a}{c}+\frac{a+b}{a}}\cdot\left(\sqrt{\frac{c^3+a^3}{c^3}}+\sqrt{\frac{a^3+b^3}{a^3}}\right)$$

$$+\frac{\frac{c+a}{c}}{\frac{a+b}{a}+\frac{b+c}{b}}\cdot\left(\sqrt{\frac{a^3+b^3}{a^3}}+\sqrt{\frac{b^3+c^3}{b^3}}\right)$$

$$=\frac{x}{y+z}(B'+C')+\frac{y}{z+x}(C'+A')+\frac{z}{x+y}(A'+B')$$

$$\left(x=\frac{a+b}{a},y=\frac{b+c}{b},z=\frac{c+a}{c},A'=\sqrt{\frac{a^3+b^3}{a^3}},B'=\sqrt{\frac{b^3+c^3}{b^3}},C'=\sqrt{\frac{c^3+a^3}{c^3}}\right)$$

$$\overset{\text{Oppenheim}}{\geq} 4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}A'B'}.\frac{\sqrt{3}}{2}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(\left(\sqrt{\frac{a^3+b^3}{a^3}}\right)\left(\sqrt{\frac{b^3+c^3}{b^3}}\right)\right)}\overset{A-G}{\geq} 3.\sqrt[6]{\frac{(a^3+b^3)(b^3+c^3)(c^3+a^3)}{a^3b^3c^3}}\overset{\text{Cesaro}}{\geq}$$

$$3.\sqrt[6]{8}=3\sqrt{2}\ \forall\ a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$