ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{(b+c)^3 \cdot \frac{c^3+a^3}{a^3+b^3} + (c+a)^3 \cdot \frac{b^3+c^3}{a^3+b^3}}{\left(\frac{b+c}{a+b}\right)^2 \cdot (c^3+a^3) + \left(\frac{c+a}{a+b}\right)^2 \cdot (b^3+c^3)} \geq \frac{36abc}{a^3+b^3+c^3}$$

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$\forall A', B', C' > 0, (A'+B'), (B'+C'), (C'+A')$ form sides of a triangle

$(\because (A'+B') + (B'+C') > (C'+A')$ and analogs$) \Rightarrow \sqrt{A'+B'}, \sqrt{B'+C'}, \sqrt{C'+A'}$

form sides of a Δ with area F (say) and $16F^2 =$

$$2\sum_{cyc}(A'+B')(B'+C') - \sum_{cyc}(A'+B')^2$$

$$= 2\sum_{cyc}\left(\sum_{cyc} A'B' + B'^2\right) - 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'B'$$

$$= 6\sum_{cyc} A'B' + 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'B' \Rightarrow 4F = 2\sqrt{\sum_{cyc} A'B'} \rightarrow (1)$$

$$\text{Now}, \forall x, y, z > 0, \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc} \frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} =$$

$$\frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x} \overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \rightarrow \text{true}$$

$$\therefore \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

$$\text{We have}: \frac{(b+c)^3 \cdot \frac{c^3+a^3}{a^3+b^3} + (c+a)^3 \cdot \frac{b^3+c^3}{a^3+b^3}}{\left(\frac{b+c}{a+b}\right)^2 \cdot (c^3+a^3) + \left(\frac{c+a}{a+b}\right)^2 \cdot (b^3+c^3)} +$$

$$\frac{(a+b)^3\cdot\frac{c^3+a^3}{b^3+c^3}+(c+a)^3\cdot\frac{a^3+b^3}{b^3+c^3}}{\left(\frac{a+b}{b+c}\right)^2\cdot(c^3+a^3)+\left(\frac{c+a}{b+c}\right)^2\cdot(a^3+b^3)}+\frac{(a+b)^3\cdot\frac{b^3+c^3}{c^3+a^3}+(b+c)^3\cdot\frac{a^3+b^3}{c^3+a^3}}{\left(\frac{a+b}{c+a}\right)^2\cdot(b^3+c^3)+\left(\frac{b+c}{c+a}\right)^2\cdot(a^3+b^3)}$$

$$=\frac{\frac{(a+b)^2}{a^3+b^3}\cdot\left(\frac{(b+c)^3}{b^3+c^3}+\frac{(c+a)^3}{c^3+a^3}\right)}{\frac{(b+c)^2}{b^3+c^3}+\frac{(c+a)^2}{c^3+a^3}}+\frac{\frac{(b+c)^2}{b^3+c^3}\cdot\left(\frac{(a+b)^3}{a^3+b^3}+\frac{(c+a)^3}{c^3+a^3}\right)}{\frac{(a+b)^2}{a^3+b^3}+\frac{(c+a)^2}{c^3+a^3}}+\frac{\frac{(c+a)^2}{c^3+a^3}\cdot\left(\frac{(a+b)^3}{a^3+b^3}+\frac{(b+c)^3}{b^3+c^3}\right)}{\frac{(a+b)^2}{a^3+b^3}+\frac{(b+c)^2}{b^3+c^3}}$$

$$=\frac{x}{y+z}(B'+C')+\frac{y}{z+x}(C'+A')+\frac{z}{x+y}(A'+B')$$

$$\left(\begin{array}{l} x=\frac{(a+b)^2}{a^3+b^3},\, y=\frac{(b+c)^2}{b^3+c^3},\, z=\frac{(c+a)^2}{c^3+a^3}, \\ A'=\frac{(a+b)^3}{a^3+b^3},\, B'=\frac{(b+c)^3}{b^3+c^3},\, C'=\frac{(c+a)^3}{c^3+a^3} \end{array}\right)$$

$$\overset{\text{Oppenheim}}{\geq} 4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via }(1)\text{ and }(2)}{\geq} 2\sqrt{\sum_{cyc}A'B'}\cdot\frac{\sqrt{3}}{2}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{(a+b)^3}{a^3+b^3},\frac{(b+c)^3}{b^3+c^3}\right)}\overset{A-G}{\geq} 3.\sqrt[6]{\frac{(a+b)^6(b+c)^6(c+a)^6}{(a^3+b^3)^2(b^3+c^3)^2(c^3+a^3)^2}}\overset{\text{Cesaro}}{\geq}$$

$$3.8abc.\frac{1}{\sqrt[3]{(a^3+b^3)(b^3+c^3)(c^3+a^3)}}\overset{A-G}{\geq} 24abc.\frac{3}{2(a^3+b^3+c^3)}=\frac{36abc}{a^3+b^3+c^3}$$

$$\forall\, a,b,c>0,'' ='' \text{ iff } a=b=c \text{ (QED)}$$