ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that :

$$\sum_{cyc} \frac{a^{2024}\left(ab^{2025} + c^{2026}\right)}{ab^{2025} + bc^{2025}} \geq 3(abc)^{\frac{2024}{3}}$$

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$\forall A', B', C' > 0, (A' + B'), (B' + C'), (C' + A')$ form sides of a triangle

$(\because (A' + B') + (B' + C') > (C' + A')$ and analogs$) \Rightarrow \sqrt{A' + B'}, \sqrt{B' + C'}, \sqrt{C' + A'}$

form sides of a Δ with area F (say) and

$$16F^2 = 2\sum_{cyc}(A' + B')(B' + C') - \sum_{cyc}(A' + B')^2$$

$$= 2\sum_{cyc}\left(\sum_{cyc} A'B' + B'^2\right) - 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'B'$$

$$= 6\sum_{cyc} A'B' + 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'^2 - 2\sum_{cyc} A'B' \Rightarrow 4F = 2\sqrt{\sum_{cyc} A'B'} \rightarrow (1)$$

Now, $\forall x, y, z > 0$, $\sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc} \frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$

Via Bergstrom, LHS of $(*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} =$

$$\frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x} \overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \rightarrow \text{true}$$

$$\therefore \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

We have : $\frac{a^{2024}\left(ab^{2025} + c^{2026}\right)}{ab^{2025} + bc^{2025}} + \frac{b^{2024}\left(bc^{2025} + a^{2026}\right)}{bc^{2025} + ca^{2025}} + \frac{c^{2024}\left(ca^{2025} + b^{2026}\right)}{ca^{2025} + ab^{2025}}$

$$= \frac{\frac{a^{2024}}{b}\left(\frac{b^{2025}}{c} + \frac{c^{2025}}{a}\right)}{\frac{b^{2024}}{c} + \frac{c^{2024}}{a}} + \frac{\frac{b^{2024}}{c}\left(\frac{c^{2025}}{a} + \frac{a^{2025}}{b}\right)}{\frac{c^{2024}}{a} + \frac{a^{2024}}{b}} + \frac{\frac{c^{2024}}{a}\left(\frac{a^{2025}}{b} + \frac{b^{2025}}{c}\right)}{\frac{a^{2024}}{b} + \frac{b^{2024}}{c}}$$

$$= \frac{x}{y+z}(B' + C') + \frac{y}{z+x}(C' + A') + \frac{z}{x+y}(A' + B')$$

$$\left(x = \frac{a^{2024}}{b}, y = \frac{b^{2024}}{c}, z = \frac{c^{2024}}{a}, A' = \frac{a^{2025}}{b}, B' = \frac{b^{2025}}{c}, C' = \frac{c^{2025}}{a}\right) \overset{\text{Oppenheim}}{\geq}$$

4F. $\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{via\ (1)\ and\ (2)}{\geq} 2\sqrt{\sum_{cyc} A'B'}.\frac{\sqrt{3}}{2} = \sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{a^{2025}}{b}.\frac{b^{2025}}{c}\right)}$

$\overset{A-G}{\geq} 3.\sqrt[6]{\frac{(abc)^{4050}}{(abc)^2}} \therefore \frac{a^{2024}\left(ab^{2025}+c^{2026}\right)}{ab^{2025}+bc^{2025}} + \frac{b^{2024}\left(bc^{2025}+a^{2026}\right)}{bc^{2025}+ca^{2025}} +$

$\frac{c^{2024}\left(ca^{2025}+b^{2026}\right)}{ca^{2025}+ab^{2025}} \geq 3.(abc)^{\frac{2024}{3}}\ \forall\, a,b,c>0, ''='' \text{ iff } a=b=c\ (\text{QED})$