ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, a^5+b^5+c^5=3$ then prove that:

$$\frac{a\sqrt[3]{a(a^5+a^3b^2+b^5)^5}}{\sqrt[3]{(a^5+a^2b^3+b^5)^2}}+\frac{b\sqrt[3]{b(b^5+b^3c^2+c^5)^5}}{\sqrt[3]{(b^5+b^2c^3+c^5)^2}}+\frac{c\sqrt[3]{c(c^5+c^3a^2+a^5)^5}}{\sqrt[3]{(c^5+c^2a^3+a^5)^2}}\geq\sqrt{3(a^2+b^2+c^2)^5}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$\sum_{cyc}\frac{a\sqrt[3]{a(a^5+a^3b^2+b^5)^5}}{\sqrt[3]{(a^5+a^2b^3+b^5)^2}}=\sum_{cyc}\frac{(a^7+a^5b^2+a^2b^5)^{\frac{5}{3}}}{(a^8+a^5b^3+a^3b^5)^{\frac{2}{3}}}\geq\frac{\left(\sum_{cyc}(a^7+a^5b^2+a^2b^5)\right)^{\frac{5}{3}}}{\left(\sum_{cyc}(a^8+a^5b^3+a^3b^5)\right)^{\frac{2}{3}}}$$

$$=\sqrt[3]{\frac{\left((a^2+b^2+c^2)(a^5+b^5+c^5)\right)^5}{\left((a^3+b^3+c^3)(a^5+b^5+c^5)\right)^2}}=\sqrt[3]{\frac{27(a^2+b^2+c^2)^5}{(a^3+b^3+c^3)^2}}\geq\sqrt{3(a^2+b^2+c^2)^5},$$

the last inequality is true by Power Mean Inequality:

$$a^3+b^3+c^3\leq 3\sqrt[5]{\left(\frac{a^5+b^5+c^5}{3}\right)^3}=3.$$

Equality holds iff $a=b=c$.