ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ such that $abc=1$ and $n,m,t\in\mathbb{N}$, then prove that

$$\frac{a^t(b^{n+m}+c^{n+m})}{a^{n+m}+b^nc^m}+\frac{b^t(c^{n+m}+a^{n+m})}{b^{n+m}+c^na^m}+\frac{c^t(a^{n+m}+b^{n+m})}{c^{n+m}+a^nb^m}\ge 3$$

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Solution 1 :

By AM – GM inequality, we have

$$\sum_{cyc}\frac{a^t(b^{n+m}+c^{n+m})}{a^{n+m}+b^nc^m}\ge 3\sqrt[3]{\frac{(abc)^t(a^{n+m}+b^{n+m})(b^{n+m}+c^{n+m})(c^{n+m}+a^{n+m})}{(a^{n+m}+b^nc^m)(b^{n+m}+c^na^m)(c^{n+m}+a^nb^m)}}\overset{?}{\ge}3$$

$$\Leftrightarrow (a^{n+m}+b^{n+m})(b^{n+m}+c^{n+m})(c^{n+m}+a^{n+m})$$
$$\ge (a^{n+m}+b^nc^m)(b^{n+m}+c^na^m)(c^{n+m}+a^nb^m)$$
$$\Leftrightarrow \sum_{cyc}a^{2(n+m)}(b^{n+m}+c^{n+m})\ge\sum_{cyc}a^{2n+m}b^{n+2m}+\sum_{cyc}a^{2(n+m)}b^mc^n\qquad(*)$$

By AM – GM inequality, we have $a^nb^m\le\frac{n}{n+m}.a^{n+m}+\frac{m}{n+m}.b^{n+m}$ (and analogs), then

$$RHS_{(*)}\le\sum_{cyc}a^{n+m}b^{n+m}\left(\frac{n}{n+m}.a^{n+m}+\frac{m}{n+m}.b^{n+m}\right)$$
$$+\sum_{cyc}a^{2(n+m)}\left(\frac{n}{n+m}.c^{n+m}+\frac{m}{n+m}.b^{n+m}\right)=\sum_{cyc}a^{2(n+m)}(b^{n+m}+c^{n+m})=LHS_{(*)},$$

which completes the proof. Equality holds iff $a=b=c=1$.

Solution 2 :

By Hölder's inequality, we have

$$a^{n+m}+b^nc^m\le\sqrt[n+m]{(a^{n+m}+b^{n+m})^n(a^{n+m}+c^{n+m})^m}\ \text{(and analogs)}$$

Then

$$\sum_{cyc}\frac{a^t(b^{n+m}+c^{n+m})}{a^{n+m}+b^nc^m}\ge\sum_{cyc}\frac{a^t(b^{n+m}+c^{n+m})}{\sqrt[n+m]{(a^{n+m}+b^{n+m})^n(a^{n+m}+c^{n+m})^m}}$$
$$\overset{AM-GM}{\ge}3\sqrt[3]{\prod_{cyc}\frac{a^t(b^{n+m}+c^{n+m})}{\sqrt[n+m]{(a^{n+m}+b^{n+m})^n(a^{n+m}+c^{n+m})^m}}}=3,$$

as desired. Equality holds iff $a=b=c=1$.