ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c,d,e>0$ then:

$$\frac{a+b+c}{d+e}+\frac{b+c+d}{e+a}+\frac{c+d+e}{a+b}+\frac{d+e+a}{b+c}+\frac{e+a+b}{c+d}\geq\frac{15}{2}$$

(Vasic's variant)

Proposed by Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania

$$\frac{a+b+c}{d+e}+1+\frac{b+c+d}{e+a}+1+\frac{c+d+e}{a+b}+1+$$

$$+\frac{d+e+a}{b+c}+1+\frac{e+a+b}{c+d}+1\geq\frac{15}{2}+5$$

$$\frac{a+b+c+d+e}{d+e}+\frac{a+b+c+d+e}{e+a}+\frac{a+b+c+d+e}{a+b}+$$

$$+\frac{a+b+c+d+e}{b+c}+\frac{a+b+c+d+e}{c+d}\geq\frac{15+10}{2}=\frac{25}{2}$$

$$(a+b+c+d+e)\left(\frac{1}{d+e}+\frac{1}{e+a}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}\right)\geq\frac{25}{2}$$

$$(a+b+c+d+e)\left(\frac{1}{d+e}+\frac{1}{e+a}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}\right)\geq$$

$$\overset{Bergstrom}{\geq}(a+b+c+d+e)\cdot\frac{(1+1+1+1+1)^2}{d+e+e+a+a+b+b+c+c+d}=$$

$$=\frac{25(a+b+c+d+e)}{2a+2b+2c+2d+2e}=\frac{25(a+b+c+d+e)}{2(a+b+c+d+e)}=\frac{25}{2},\text{ true}$$

$$\Rightarrow\frac{a+b+c}{d+e}+\frac{b+c+d}{e+a}+\frac{c+d+e}{a+b}+\frac{d+e+a}{b+c}+\frac{e+a+b}{c+d}\geq\frac{15}{2},(\forall)a,b,c,d,e>0$$

Equality is if $a=b=c=d=e$.

Solution 2 by Tapas Das-India

Let $(a+b+c+d+e)=x$

$$\therefore LHS=\frac{a+b+c}{x-(a+b+c)}+\frac{b+c+d}{x-(b+c+d)}+\frac{c+d+e}{x-(c+d+e)}+$$

$$+\frac{d+e+a}{x-(d+e+a)}+\frac{e+a+b}{x-(e+a+b)}$$

Let $f(p)=\frac{p}{x-p},p>0$

$$\therefore f'(p)=\frac{x}{(x-p)^2}\ \therefore f''(p)=\frac{2x}{(x-p)^3}>0$$

$\therefore f$ **is convex, using Jensen inequality**

$$f(a+b+c)+f(b+c+d)+f(c+d+e)+f(d+e+a)+f(e+a+b)\geq$$

$$\geq 5f\left(\frac{3a+3b+3c+3d+3e}{5}\right)=5f\left(\frac{3x}{5}\right)=5\cdot\frac{\frac{3x}{5}}{x-\frac{3x}{5}}=5\times\frac{3}{2}=\frac{15}{2}$$

Solution 3 by Sakthi Vel-India

$$\frac{a+b+c}{d+e}=\frac{a+b+c}{d+e}+\frac{d+e}{d+e}-1=\frac{a+b+c+d+e}{d+e}-1$$

$$\sum_{cyc}\frac{a+b+c}{d+e}=(a+b+c+d+e)\sum_{cyc}\frac{1}{d+e}-5$$

$$\geq(a+b+c+d+e)\frac{(1+1+1+1+1)^2}{2(a+b+c+d+e)}-5=\frac{25}{2}-5=\frac{15}{2}$$