ROMANIAN MATHEMATICAL MAGAZINE

If $x, y \in \left[0, \frac{\pi}{2}\right)$, then :

$$(\sin x+\sin y)(x+y)(\tan x+\tan y) \overset{(*)}{\leq} (\sin x+y)(x+\tan y)(\tan x+\sin y)$$
$$\overset{(**)}{\leq} (\sin x+\tan y)(x+y)(\tan x+\sin y)$$

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$$\text{If } x=y=0, \text{then : LHS of } (*) = \text{RHS of } (*) = 0 \text{ and LHS of } (**)$$
$$= \text{RHS of } (**) = 0$$
$$\text{If } x=0, y\in\left(0,\frac{\pi}{2}\right), \text{then : LHS of } (*) = \text{RHS of } (*) = y\sin y\tan y \text{ and LHS of } (**)$$
$$= \text{RHS of } (**) = y\sin y\tan y$$
$$\text{If } y=0, x\in\left(0,\frac{\pi}{2}\right), \text{then : LHS of } (*) = \text{RHS of } (*) = x\sin x\tan x \text{ and LHS of } (**)$$
$$= \text{RHS of } (**) = x\sin x\tan x$$
$$\text{We now consider } x,y\in\left(0,\frac{\pi}{2}\right) \text{ and}$$
$$(\sin x+\sin y)(x+y)(\tan x+\tan y) \leq (\sin x+y)(x+\tan y)(\tan x+\sin y)$$
$$\Leftrightarrow \frac{\tan x+\tan y}{\tan x+\sin y} - 1 \leq \frac{(\sin x+y)(x+\tan y)}{(\sin x+\sin y)(x+y)} - 1$$
$$\Leftrightarrow \frac{\tan y - y + y - \sin y}{\tan x+\sin y} \leq$$
$$\frac{x\sin x+\sin x\tan y+xy+y\tan y-x\sin x-y\sin x-x\sin y-y\sin y}{(\sin x+\sin y)(x+y)}$$
$$\Leftrightarrow \frac{(\tan y-y)\sin x + x(y-\sin y) + y\big((\tan y-y)+(y-\sin y)\big)}{(\sin x+\sin y)(x+y)}$$
$$\geq \frac{(\tan y-y)+(y-\sin y)}{\tan x+\sin y}$$
$$\Leftrightarrow (\tan y-y)\left(\frac{y+\sin x}{(\sin x+\sin y)(x+y)} - \frac{1}{\tan x+\sin y}\right)$$
$$+(y-\sin y)\left(\frac{1}{\sin x+\sin y} - \frac{1}{\tan x+\sin y}\right) \geq 0 \Leftrightarrow$$
$$\begin{pmatrix}\tan y\\ -y\end{pmatrix}\left(\frac{\tan x\sin x+y\tan x+\sin x\sin y+y\sin y-x\sin x-x\sin y-y\sin x-y\sin y}{(\sin x+\sin y)(x+y)(\tan x+\sin y)}\right)$$
$$+(y-\sin y)\left(\frac{\tan x-\sin x}{(\sin x+\sin y)(\tan x+\sin y)}\right) \geq 0$$
$$\Leftrightarrow \begin{pmatrix}\tan y\\ -y\end{pmatrix}\left(\frac{\sin x(\tan x-x)+y\big((\tan x-x)+(x-\sin x)\big)-\sin y(x-\sin x)}{(\sin x+\sin y)(x+y)(\tan x+\sin y)}\right)$$
$$+(y-\sin y)\left(\frac{\tan x-\sin x}{(\sin x+\sin y)(\tan x+\sin y)}\right) \geq 0$$
$$\Leftrightarrow (\tan y-y)\left(\frac{(\tan x-x)(\sin x+y)+(y-\sin y)(x-\sin x)}{(\sin x+\sin y)(x+y)(\tan x+\sin y)}\right)$$

$$+(y-\sin y)\left(\frac{\tan x-\sin x}{(\sin x+\sin y)(\tan x+\sin y)}\right)\geq 0$$

$$\rightarrow \text{true} \because \tan x > x > \sin x \text{ and } \tan y > y > \sin y \ \forall x, y \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore (\sin x+\sin y)(x+y)(\tan x+\tan y) < (\sin x+y)(x+\tan y)(\tan x+\sin y)$$

and combining *all* cases,

$$\boxed{\boxed{(\sin x+\sin y)(x+y)(\tan x+\tan y) \leq (\sin x+y)(x+\tan y)(\tan x+\sin y)\ \forall x, y \in \left[0, \frac{\pi}{2}\right)}},$$

$$'' ='' \text{ iff } (x=y=0) \text{ or } \left(x=0, y \in \left(0, \frac{\pi}{2}\right)\right) \text{ or } \left(y=0, x \in \left(0, \frac{\pi}{2}\right)\right)$$

Again, $\boxed{(\sin x+y)(x+\tan y)(\tan x+\sin y) \leq (\sin x+\tan y)(x+y)(\tan x+\sin y)}$

$$\Leftrightarrow \frac{\sin x+\tan y}{\sin x+y}-1 \geq \frac{x+\tan y}{x+y}-1 \Leftrightarrow \frac{\tan y-y}{\sin x+y} \geq \frac{\tan y-y}{x+y}$$

$$\Leftrightarrow (\tan y-y)\left(\frac{x+y-\sin x-y}{(x+y)(\sin x+y)}\right) \geq 0 \boxed{\Leftrightarrow \frac{(\tan y-y)(y-\sin y)}{(x+y)(\sin x+y)} \geq 0} \rightarrow \text{true}$$

$$\because \tan y > y > \sin y \ \forall x, y \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore (\sin x+y)(x+\tan y)(\tan x+\sin y) < (\sin x+\tan y)(x+y)(\tan x+\sin y)$$

and combining *all* cases,

$$\boxed{\boxed{(\sin x+y)(x+\tan y)(\tan x+\sin y) \leq (\sin x+\tan y)(x+y)(\tan x+\sin y)\ \forall x, y \in \left[0, \frac{\pi}{2}\right)}},$$

$$'' ='' \text{ iff } (x=y=0) \text{ or } \left(x=0, y \in \left(0, \frac{\pi}{2}\right)\right) \text{ or } \left(y=0, x \in \left(0, \frac{\pi}{2}\right)\right) \text{ (QED)}$$