ROMANIAN MATHEMATICAL MAGAZINE

If $0 \leq x \leq \frac{\pi}{2}$ then:

$$\arcsin(\cos^2 x) \leq \frac{2\cos^2 x}{1+\sin x}$$

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Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by David Chatarasvili-Georgia

Solution 1 by Ravi Prakash-New Delhi-India

$$\sin^{-1}(\cos^2 x) \leq \frac{2\cos^2 x}{1+\sin x}$$

$$\Leftrightarrow \sin^{-1}(1-\sin^2 x) \leq 2(1-\sin x) \Leftrightarrow \sin^{-1}(1-t^2) \leq 2(1-t)$$

Let $f(t) = \sin^{-1}(1-t^2) - 2(1-t)$

$$f'(t) = \frac{-2t}{\sqrt{1-(1-t^2)^2}} + 2 = \frac{-2t}{\sqrt{(2-t^2)t^2}} + 2 = 2\left[1 - \frac{1}{\sqrt{2-t^2}}\right] > 0 \text{ for } 0 < t < 1$$

$\Rightarrow f(t)$ increases on $[0,1]$

$\Rightarrow f(t) \leq f(1) = 0$ for $0 \leq t \leq 1 \Rightarrow \sin^{-1}(1-t^2) \leq 2(1-t)$ for $0 \leq t \leq 1$

Solution 2 by David Chatarasvili-Georgia

Consider the function: $f(x) = \arcsin(\cos^2 x) \cdot \frac{2\cos^2 x}{1+\sin x}$

$$x \in \left[0, \frac{\pi}{2}\right]$$

$$f'(x) = \frac{-2\cos x \sin x}{\sqrt{1-\cos^4 x}} - \frac{-4\cos x \sin x\,(1+\sin x) - \cos x\,(2\cos^2 x)}{(1+\sin x)^2} =$$

$$= \frac{2\cos x\,(2\sin x\,(1+\sin x) + \cos^2 x)}{(1+\sin x)^2} - \frac{2\sin x \cdot \cos x}{\sqrt{1-\cos^4 x}} =$$

$$= \frac{2\cos(2\sin x + 2\sin^2 x + 1 - \sin^2 x)}{(1+\sin x)^2} - \frac{2\sin x \cdot \cos x}{\sqrt{1-\cos^4 x}} =$$

$$= 2\cos x\left(1 - \frac{\sin x}{\sqrt{1-\cos^4 x}}\right) = \frac{2\cos x\left(\sqrt{1-\cos^4 x} - \sin x\right)}{\sqrt{1-\cos^4 x}}$$

$$f'(x) = 0 \Rightarrow \left[\begin{array}{l} \cos x = 0 \\ \sqrt{1-\cos^4 x} = \sin x \quad (2) \end{array}\right.$$

$$(1) \Rightarrow x = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}, x \in \left[0, \frac{\pi}{2}\right] \Rightarrow x = \frac{\pi}{2}$$

$$(2) \Leftrightarrow 1 - \cos^4 x = \sin x \left(\sin x \geq 0; x \in \left[0, \frac{\pi}{2}\right]\right)$$

$$1 - (1 - \sin^2 x)^2 = 1 - (1 - 2\sin^2 x + \sin^4 x) = 2\sin^2 x - \sin^4 x =$$

$$= \sin^2 x \Rightarrow \sin^2 x - \sin^4 x = 0 \Rightarrow \sin^2 x\,(1 - \sin^2 x) = 0 \Rightarrow$$

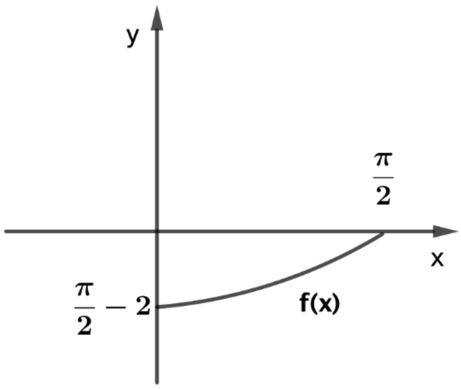
$$\Rightarrow \sin^2 x \cdot \cos^2 x = 0 \Rightarrow \left[\begin{array}{l}\sin x = 0\\ \cos x = 0\end{array}\right. \Rightarrow \sin x = 0$$

$$x = \pi k, k \in \mathbb{Z}; x \in \left[0, \frac{\pi}{2}\right] \Rightarrow x = 0$$

$$\begin{cases} f'(x) = 0 \Rightarrow x = 0, x = \frac{\pi}{2} \\ f'(x) \geq 0; \forall x \in \left[0, \frac{\pi}{2}\right] \end{cases} \Rightarrow f(x) \text{ strictly increasing function}$$

$$f(0) = \arcsin 1 - \frac{2}{1} = \frac{\pi}{2} - 2 < 0$$

$$f\left(\frac{\pi}{2}\right) = \arcsin 0 - 0 = 0$$



$$\max_{x \in \left[0, \frac{\pi}{2}\right]} f(x) = f\left(\frac{\pi}{2}\right) = 0 \Rightarrow \forall x \in \left[0, \frac{\pi}{2}\right]\ f(x) \leq 0 \Rightarrow \arcsin(\cos^2 x) - \frac{2\cos^2 x}{1 + \sin x} \leq 0$$

$$\Rightarrow \arcsin(6\cos^2 x) \leq \frac{2\cos^2 x}{1 + \sin x}; \forall x \in \left[0, \frac{\pi}{2}\right]$$