ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 1, m, n, p \geq 1$ then:

$$(x-1)^{m+n+p} + x^{m+n} + x^{n+p} + x^{p+m} + 1 \leq x^{m+n+p} + x^m + x^n + x^p$$

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We first show that

$$x^m + 1 \leq (x+1)^m \quad \forall x > 0, m \geq 1$$

Let $f(x) = x^m + 1 - (x+1)^m, x \geq 0 \Rightarrow f'(x) = mx^{m-1} - m(x+1)^{m-1} \leq 0, x > 0$

$$\Rightarrow f(x) \text{ decreases on } [0, \infty)$$

Thus, $f(x) \leq f(0) \quad \forall x > 0 \Rightarrow x^m + 1 - (x+1)^m \leq 0; \forall x > 0$

$$\Rightarrow (x-1)^m + 1 \leq x^m; \forall x > 1 \Rightarrow (x-1)^m \leq x^m - 1; \forall x > 1, m \geq 1$$

Thus, for $x > 1, m, n, p \geq 1$

$$(x-1)^m (x-1)^n (x-1)^p \leq (x^m - 1)(x^n - 1)(x^p - 1)$$

$$\Rightarrow (x-1)^{m+n+p} \leq x^{m+n+p} - x^{m+n} - x^{m+p} - x^{n+p} + x^m + x^n + x^p - 1$$

$$\Rightarrow (x-1)^{m+n+p} + x^{m+n} + x^{m+p} + x^{n+p} + 1 \leq x^{m+n+p} + x^m + x^n + x^p$$