ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ then:

$$e(x^x + y^y + z^z) \geq 3\sqrt[3]{e^{x+y+z}}$$

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Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by Hikmat Mammadov-Azerbaijan, Solution 3 by Khaled Abd Imouti-Damascus-Syria, Solution 4 by Pin Reak Smey-Indonesia

Solution 1 by Ravi Prakash-New Delhi-India

Let $f(x) = x(\ln x - 1) + 1; \forall x > 0$

$$f'(x) = \ln x - 1 + x\left(\frac{1}{x}\right) = \ln x \Rightarrow f'(x) < 0 \text{ for } 0 < x < 1, > 0 \text{ if } x > 1$$

Thus, for $0 < x \leq 1, f(1) \leq f(x)$ and for $x \geq 1, f(1) \leq f(x)$

$$\Rightarrow f(x) \geq f(1) = 0; \forall x > 0 \Rightarrow x(\ln x - 1) + 1 \geq 0; \forall x > 0$$

$$\Rightarrow x \ln x \geq x - 1 \Rightarrow \ln(x^x) \geq x - 1 \Rightarrow x^x \geq e^{x-1}; \forall x > 0 \Rightarrow ex^x \geq e^x$$

Thus, for $x, y, z > 0$

$$e(x^x + y^y + z^z) \geq e^x + e^y + e^z \geq 3(e^x e^y e^z)^{\frac{1}{3}} = 3(e^{x+y+z})^{\frac{1}{3}}$$

$$\Rightarrow e(x^x + y^y + z^z) \geq 3\left(e^{\frac{x+y+z}{3}}\right)$$

Equality when $x = y = z = 1$.

Solution 2 by Hikmat Mammadov-Azerbaijan

The inequality of arithmetic and geometric means gives $\sqrt[3]{e^x e^y e^z} \leq \frac{e^x + e^y + e^z}{3}$

So, $3\sqrt[3]{e^{x+y+z}} \leq e^x + e^y + e^z$

The study of variations of the function $f: x \to 1 + x\ln(x) - x$

Shows that the minimum of this function on, $\mathbb{R}^{+*}$ is 0 (for $x = 1$)

So, $x \leq 1 + x\ln(x) - x, y \leq 1 + y\ln(y) - y$ and $z \leq 1 + z\ln(z) - z$

So, $e^x \leq e \cdot x^x, e^y \leq e \cdot y^y$ and $e^z \leq e \cdot z^z$

So, $e^x + e^y + e^z \leq e \cdot (x^x + y^y + z^z)$

Finally: $3 \cdot \sqrt[3]{e^{x+y+z}} \leq e \cdot (x^x + y^y + z^z)$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$e\cdot(x^x+y^y+z^z)\geq 3\cdot\left(e^{\frac{x+y+z}{3}}\right)$$

$$e\cdot\left(e^{x\ln(x)}+e^{y\ln(y)}+e^{z\ln(z)}\right)\geq 3\left(e^{\frac{x+y+z}{3}}\right)$$

$$e^{1+x\ln(x)}+e^{1+y\ln(y)}+e^{1+3\ln(z)}\geq 3\cdot\left(e^{\frac{x+y+z}{3}}\right)$$

$$f(x)=1+x\ln(x),'f(x)=\ln(x)+1\Rightarrow ''f(x)=\frac{1}{x}>0$$

$$l_1\geq 3\cdot e^{1+\left(\frac{x+y+z}{3}\right)\ln\left(\frac{x+y+z}{3}\right)},\qquad l_1\geq e\cdot\left(\frac{x+y+z}{3}\right)$$

$$1+x\ln(x)\overset{?}{\geq} x,\qquad x\ln(x)\geq x-1,\qquad x\ln(x)-x+1\overset{?}{\geq}0$$

$$f(x)=x\ln(x)-x+1,x>0$$

$$\lim_{x\to 0^+}f(x)=1,\ \lim_{x\to+\infty}f(x)=+\infty$$

$$'f(x)=\ln(x)+1-1=\ln(x)$$

x	0 $\qquad$ 1 $\qquad$ $+\infty$
$f'(x)$	$--------0++++++++$
$f(x)$	↘ ↗

Solution 4 by Pin Reak Smey-Indonesia

We have $ex^x\geq e^x\Leftrightarrow \ln e+\ln x^x\geq x\Leftrightarrow 1+x\ln x-x\geq 0$

Let $f(x)=1+x\ln x-x,\ f'(x)=\ln x$

x	0 $\qquad$ 1 $\qquad$ $+\infty$
$f'(x)$	$--------0++++++++$
$f(x)$	↘ ↗

$$\Rightarrow f(x)\geq 0\Rightarrow f(1)=0$$

$$ex^x\geq e^x,\forall x>0\Rightarrow\sum_{cyc}ex^x\geq\sum_{cyc}e^x$$

But $\sum_{cyc}e^x\geq 3\sqrt[3]{e^{x+y+z}}$, thus $\sum_{cyc}ex^x\geq 3\sqrt[3]{e^{x+y+z}}$