ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z\in\mathbb{R}$ then:

$$2(5-x-y-z)^2+14(x^2+y^2+z^2)\geq 35$$

Proposed by Daniel Sitaru – Romania

Solution 1 by George Florin Șerban-Romania

$$x^2+y^2+z^2\overset{CBS}{\geq}\frac{(x+y+z)^2}{3}$$

$$S=x+y+z$$

$$\Rightarrow 2\cdot(5-S)^2+14\sum_{cyc}x^2\geq 2(5-S)^2+\frac{14S^2}{3}\geq 35\Rightarrow$$

$$\Rightarrow 50-20S+2S^2+\frac{14S^2}{3}\geq 35\Rightarrow 150-60S+6S^2+14S^2\geq 105\Rightarrow$$

$$\Rightarrow 20S^2-60S+45\geq 0|:5$$

$$\Rightarrow 4S^2-12S+9\geq 0\Rightarrow (2S-3)^2\geq 0$$

true, $(\forall)S\in\mathbb{R}$

Then $2\cdot(S-x-y-z)^2+14(x^2+y^2+z^2)\geq 35,(\forall)x,y,z\in\mathbb{R}$

Solution 2 by George Florin Șerban-Romania

The second method:

$$2\cdot(5-x-y-z)^2+14(x^2+y^2+z^2)\geq 35$$

$$2\cdot(25-10\sum_{cyc}x+(x+y+z)^2+14\sum_{cyc}x^2\geq 35$$

$$50-20\sum_{cyc}x+2\sum_{cyc}x^2+4\sum_{cyc}xy+14\sum_{cyc}x^2-35\geq 0$$

$$16\sum_{cyc}x^2+4\sum_{cyc}xy-20\sum_{cyc}x+15\geq 0$$

$$\left(x+2y-\frac{3}{2}\right)^2+\left(y+2z-\frac{3}{2}\right)^2+\left(z+2x-\frac{3}{2}\right)^2+$$

$$+11\left(x-\frac{1}{2}\right)^2+11\left(y-\frac{1}{2}\right)^2+11\left(z-\frac{1}{2}\right)^2=$$

$$=x^2+4y^2+\frac{9}{4}+4xy-6y-3x+y^2+4z^2+\frac{9}{4}+$$

$$+11x^2-11x+\frac{11}{4}+11y^2-11y+\frac{11}{y}+11z^2-11z+\frac{11}{4}=$$

$$=16\sum_{cyc}x^2+4\sum_{cyc}xy-20\sum_{cyc}x+15\ge 0$$

true, because $\left(x+2y-\frac{3}{2}\right)^2\ge 0,(\forall)x,y\in\mathbb{R}$

$$\left(y+2z-\frac{3}{2}\right)^2\ge 0;(\forall)y,z\in\mathbb{R},\left(z+2x-\frac{3}{3}\right)^2\ge 0$$

$(\forall)y\in\mathbb{R},\left(z-\frac{1}{2}\right)^2\ge 0;(\forall)z\in\mathbb{R}$. Equality is if $x-\frac{1}{2}=y-\frac{1}{2}=z-\frac{1}{2}=0$

$$x+2y-\frac{3}{2}=y+2z-\frac{3}{2}=z+2x-\frac{3}{2}=0\Rightarrow x=y=z=\frac{1}{2}\in\mathbb{R}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$2(5-x-y-z)^2+14(x^2+y^2+z^2)\ge 35$$

$$\Leftrightarrow 2[25+x^2+y^2+z^2-10x-10y-10z+2xy+2yz+2zx]+$$

$$+14x^2+14y^2+14z^2\ge 35$$

$$\Leftrightarrow 16x^2+16y^2+16z^2+4xy+4yz+4zx-$$

$$-20x+20y-20z+15\ge 0$$

$$\Leftrightarrow 3(4x^2-4x+1)+3(4y^2-4y+1)+3(4z^2-4z+1)+$$

$$+2(1+x^2+y^2+2xy-2x-2y)+$$

$$+2(1+y^2+z^2+2yz-2y-2z)+$$

$$+2(1+z^2+x^2+2zx-2z-2x)\ge 0\Leftrightarrow 3[(2x-1)^2+(2y-1)^2+(2z-1)^2]+$$

$$+2[(1-x-y)^2+(1-y-z)^2+(1-z-x)^2]\ge 0$$

which is true and equality when $x=y=z=\frac{1}{2}$.

Solution 4 by Khaled Abd Imouti-Damascus-Syria

$$2\big(5-(x+y+z)\big)^2+14(x^2+y^2+z^2)\overset{?}{\ge}35$$

$$2[25-10(x+y+z)+(x^2+y^2+z^2)+2(xy+yz+zx)]+14(x^2+y^2+z^2)\overset{?}{\ge}35$$

$$16\underbrace{(x^2+y^2+z^2)}_{A}+4\underbrace{(xy+yz+zx)}_{B}-20\underbrace{(x+y+z)}_{\gamma}+15\overset{?}{\ge}0$$

$$P_1=16A+4B-20\gamma+15\overset{?}{\ge}0$$

but: $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$

$\gamma^2 = A + 2B$, but $B \le A$ (Cauchy Schwarz inequality)

$$\gamma^2 \le 3A, A \ge \frac{1}{3}\gamma^2, B = \frac{\gamma^2 - A}{2}$$

$$P_1 = 16A + 4\left(\frac{\gamma^2 - A}{2}\right) - 20\gamma + 15 \overset{?}{\ge} 0$$

$$P_1 = 16A + 2(\gamma^2 - A) - 20\gamma + 15 \overset{?}{\ge} 0$$

$$P_1 = 2\gamma^2 - 20\gamma + 14A + 15 \overset{?}{\ge} 0$$

$$P_1 = 2\left(\gamma^2 - 10\gamma + 7A + \frac{15}{2}\right) \overset{?}{\ge} 0$$

$A \ge \frac{1}{3}\gamma^2$. So: $l_1 \ge 2\left(\gamma^2 - 10\gamma + \frac{7}{3}\gamma^2 + \frac{15}{2}\right)$, $l_1 \ge 2\left(\frac{10}{3}\gamma^2 - 10\gamma + \frac{15}{2}\right)$

$\Delta = 100 - 4\left(\frac{10}{3}\right)\left(\frac{15}{2}\right) = 100 - 100 = 0$.So: $\frac{10}{3}\gamma^2 - 10\gamma + \frac{15}{2} \ge 0, \forall\gamma \in \mathbb{R}$

and then: $l_1 \ge 0$