ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\sqrt[3]{4a}+\sqrt[3]{9b}+\sqrt[3]{25c}\le\sqrt[3]{100(a+b+c)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by George Florin Șerban – Romania

$$\sqrt[3]{4a}+\sqrt[3]{9b}+\sqrt[3]{25c}=\sqrt[3]{2}\cdot\sqrt[3]{2}\cdot\sqrt[3]{a}+\sqrt[3]{3}\cdot\sqrt[3]{3}\cdot\sqrt[3]{b}+\sqrt[3]{5}\cdot\sqrt[3]{5}\cdot\sqrt[3]{c}\overset{Holder}{\le}$$

$$\le\left(\sqrt[3]{2}^3+\sqrt[3]{3}^3+\sqrt[3]{5}^3\right)^{\frac{1}{3}}\cdot\left(\sqrt[3]{2}^3+\sqrt[3]{3}^3+\sqrt[3]{5}^3\right)^{\frac{1}{3}}\cdot\left(\sqrt[3]{a}^3+\sqrt[3]{b}^3+\sqrt[3]{c}^3\right)^{\frac{1}{3}}=$$

$$=(2+3+5)^{\frac{1}{3}}\cdot(2+3+5)^{\frac{1}{3}}\cdot(a+b+c)^{\frac{1}{3}}=$$

$$=\sqrt[3]{10\cdot10\cdot(a+b+c)}=\sqrt[3]{100(a+b+c)}$$

then

$$\sqrt[3]{4a}+\sqrt[3]{9b}+\sqrt[3]{25c}\le\sqrt[3]{100(a+b+c)}$$

$(\forall)a, b, c > 0$, true

Solution 2 by Michel Sterghiou – Greece

$$\sqrt[3]{4a}+\sqrt[3]{9b}+\sqrt[3]{25c}\le\sqrt[3]{100(a+b+c)}\quad(1)$$

Let $a=2x, b=3y, c=5z$ (1) $\rightarrow 2\sqrt[3]{x}+3\sqrt[3]{y}+5\sqrt[3]{z}\le\sqrt[3]{100(2x+3y+5z)}$ (2)

Using the fact that if $(t)=\sqrt[3]{t}\ \ (0,+\infty)$ is concave we have by generalize Jensen

LHS (2) $\le(2+3+5)\cdot\sqrt[3]{\frac{2x+3y+5z}{2+3+5}}=\sqrt[3]{1000\cdot\frac{a+b+c}{10}}=\sqrt[3]{100(a+b+c)}=RHS$ of (2).

Equality for $a=2, b=3, c=5$.