If $x \in \left(0, \frac{\pi}{4}\right)$, then :

$$2\sin x + \tan x < 3x + (\ln x).\ln(1-x)$$

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$$\forall x \in \left(0, \frac{\pi}{4}\right), \tan x \overset{?}{<} \frac{4x}{3} \Leftrightarrow 3\sin x \overset{?}{<} 4x\cos x$$

$$\Leftrightarrow \frac{3}{2}.2\sin x\cos x \overset{?}{<} 2x.2\cos^2 x \Leftrightarrow \frac{3}{2}.\sin 2x \overset{?}{<} 2x.(1+\cos 2x)$$

$$\Leftrightarrow \boxed{2y(1+\cos y) \underset{(\bullet)}{\overset{?}{>}} 3\sin y} \left(0 < y = 2x < \frac{\pi}{2}\right)$$

$$\text{Let } F(y) = \sin y - y + \frac{y^3}{6} - \frac{y^5}{120} \ \forall y \in \left[0, \frac{\pi}{2}\right)$$

$$\therefore F'(y) = \cos y - \frac{y^4}{24} + \frac{y^2}{2} - 1 \text{ and } F''(y) = -\left(\sin y - \left(y - \frac{y^3}{6}\right)\right)$$

$$\text{Let } P(y) = \sin y - y + \frac{y^3}{6} \ \forall y \in \left[0, \frac{\pi}{2}\right) \therefore P'(y) = \cos y + \frac{y^2}{2} - 1 \text{ and}$$

$$P''(y) = y - \sin y \geq 0 \Rightarrow P'(y) \text{ is } \uparrow \text{ on } \left[0, \frac{\pi}{2}\right) \Rightarrow P'(y) \geq P'(0) = 0$$

$$\Rightarrow P(y) \text{ is } \uparrow \text{ on } \left[0, \frac{\pi}{2}\right) \Rightarrow P(y) \geq P(0) = 0 \Rightarrow \sin y \geq y - \frac{y^3}{6} \ \forall y \in \left[0, \frac{\pi}{2}\right)$$

$$\Rightarrow F''(y) \leq 0 \Rightarrow F'(y) \text{ is } \downarrow \text{ on } \left[0, \frac{\pi}{2}\right) \Rightarrow F'(y) \leq F'(0) = 0 \Rightarrow F(y) \text{ is } \downarrow \text{ on } \left[0, \frac{\pi}{2}\right)$$

$$\Rightarrow F(y) \leq F(0) = 0 \Rightarrow \sin y \leq y - \frac{y^3}{6} + \frac{y^5}{120} \ \forall y \in \left[0, \frac{\pi}{2}\right)$$

$$\therefore \forall y \in \left(0, \frac{\pi}{2}\right), \boxed{\sin y \overset{(1)}{>} y - \frac{y^3}{6}} \text{ and } \boxed{\sin y \overset{(2)}{<} y - \frac{y^3}{6} + \frac{y^5}{120}}$$

$$\therefore \text{via } (2), \text{RHS of } (\bullet) < 3y - \frac{y^3}{2} + \frac{y^5}{40} \overset{?}{<} 2y(1+\cos y) \Leftrightarrow \boxed{\cos y \underset{(\bullet\bullet)}{\overset{?}{>}} \frac{1}{2} - \frac{y^2}{4} + \frac{y^4}{80}}$$

$$\text{Let } f(y) = \cos y - \frac{1}{2} + \frac{y^2}{4} - \frac{y^4}{80} \ \forall y \in \left(0, \frac{\pi}{2}\right]$$

$$\therefore f'(y) = -\sin y - \frac{y^3}{20} + \frac{y}{2} \overset{\text{via }(1)}{<} -y + \frac{y^3}{6} - \frac{y^3}{20} + \frac{y}{2} = -\frac{y}{2} + \frac{7y^3}{60} = -\frac{y}{2}\left(\frac{30 - 7y^2}{30}\right)$$

$$< 0 \left(\because 30 - 7y^2 \geq 30 - \frac{7\pi^2}{4} > 0\right) \Rightarrow f(y) \text{ is } \downarrow \text{ on } \left(0, \frac{\pi}{2}\right] \Rightarrow f(y) \geq f\left(\frac{\pi}{2}\right)$$

$$= -\frac{1}{2} + \frac{\pi^2}{16} - \frac{\pi^4}{16.80} \approx .0407 > 0 \Rightarrow \cos y - \frac{1}{2} + \frac{y^2}{4} - \frac{y^4}{80} > 0 \ \forall y \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true } \therefore \forall x \in \left(0, \frac{\pi}{4}\right), \tan x < \frac{4x}{3} \Rightarrow 2\sin x + \tan x < 2x + \frac{4x}{3}$$

$$= \frac{10x}{3} \overset{?}{<} 3x + (\ln x).\ln(1-x) \Leftrightarrow \boxed{(\ln x).\ln(1-x) \underset{(*)}{\overset{?}{>}} \frac{x}{3}}$$

$$\text{Let } g(x) = -\ln(1-x) - \left(x + \frac{x^2}{2} + \frac{x^3}{3}\right) \forall x \in [0, \frac{\pi}{4}) \therefore g'(x) = \frac{x^3}{1-x} \geq 0 \Rightarrow g(x)$$

$$\text{is} \uparrow \text{on } [0, \frac{\pi}{4}) \Rightarrow g(x) \geq g(0) = 0 \Rightarrow \boxed{-\ln(1-x) \overset{(i)}{>} x + \frac{x^2}{2} + \frac{x^3}{3}} \forall x \in \left(0, \frac{\pi}{4}\right)$$

$$\text{Let } h(x) = -\ln x - \left(1 - x + \frac{(1-x)^2}{2}\right) \forall x \in (0,1] \therefore h'(x) = -\left(x + \frac{1}{x} - 2\right) \leq 0$$

$$\Rightarrow h(x) \text{ is} \downarrow \text{on } (0,1] \Rightarrow h(x) \geq h(1) = 0 \Rightarrow -\ln x \geq 1 - x + \frac{(1-x)^2}{2} \forall x \in (0,1]$$

$$\therefore \boxed{-\ln x \overset{(ii)}{>} 1 - x + \frac{(1-x)^2}{2}} \forall x \in \left(0, \frac{\pi}{4}\right) \therefore (i).(ii) \Rightarrow (\ln x).\ln(1-x)$$

$$= (-\ln x).(-\ln(1-x)) > \left(x + \frac{x^2}{2} + \frac{x^3}{3}\right)\left(1 - x + \frac{(1-x)^2}{2}\right) \overset{?}{>} \frac{x}{3}$$

$$\Leftrightarrow \boxed{\boxed{\frac{x(2x^4 - 5x^3 - 15x + 14)}{12} \underbrace{\overset{?}{>}}_{(**)} 0}}$$

$$\text{Now, } 2x^4 - 5x^3 - 15x + 14 = \frac{5x-4}{625}(250x^3 - 425x^2 - 340x - 2147) + \frac{162}{625}$$

$$\because x < \frac{\pi}{4} < \frac{4}{5} \therefore 250x^3 - 425x^2 - 340x - 2147$$

$$< 250x^2.\frac{4}{5} - 425x^2 - 340x - 2147 = -225x^2 - 340x - 2147 < 0 \text{ and}$$

$$\because 5x - 4 < 0 \therefore \frac{5x-4}{625}(250x^3 - 425x^2 - 340x - 2147) + \frac{162}{625} > 0 + \frac{162}{625} > 0$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore 2\sin x + \tan x < 3x + (\ln x).\ln(1-x) \ \forall x \in \left(0, \frac{\pi}{4}\right) \text{ (QED)}$$