ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = 1$ and $\lambda \geq \frac{2}{5}$, then :

$$\sum_{cyc} \sqrt{2\lambda x^2 + 6\lambda xy + (2\lambda + 1)y^2} \leq \sqrt{10\lambda + 1}$$

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$$\sum_{cyc} \sqrt{2\lambda x^2 + 6\lambda xy + (2\lambda + 1)y^2} \overset{?}{\leq} \sum_{cyc} \left(\left(\frac{x+y}{2} \right) . \sqrt{10\lambda + 1} \right)$$

$$\Leftrightarrow y.\frac{(t_1 + 1)}{2}.\sqrt{10\lambda + 1} + z.\frac{(t_2 + 1)}{2}.\sqrt{10\lambda + 1} + x.\frac{(t_3 + 1)}{2}.\sqrt{10\lambda + 1}$$

$$\overset{?}{\geq} y.\sqrt{2\lambda t_1^2 + 6\lambda t_1 + 2\lambda + 1} + z.\sqrt{2\lambda t_2^2 + 6\lambda t_2 + 2\lambda + 1}$$

$$+x.\sqrt{2\lambda t_3^2 + 6\lambda t_3 + 2\lambda + 1} \left(t_1 = \frac{x}{y}, t_2 = \frac{y}{z}, t_3 = \frac{z}{x} \right)$$

$$\Leftrightarrow y\left(\frac{(t_1 + 1)}{2}.\sqrt{10\lambda + 1} - \sqrt{2\lambda t_1^2 + 6\lambda t_1 + 2\lambda + 1} \right)$$

$$+z\left(\frac{(t_2 + 1)}{2}.\sqrt{10\lambda + 1} - \sqrt{2\lambda t_2^2 + 6\lambda t_2 + 2\lambda + 1} \right)$$

$$+x\left(\frac{(t_3 + 1)}{2}.\sqrt{10\lambda + 1} - \sqrt{2\lambda t_3^2 + 6\lambda t_3 + 2\lambda + 1} \right) \underset{(*)}{\overset{?}{\geq}} 0$$

Let $f(t) = \frac{(t+1)}{2}.\sqrt{10\lambda + 1} - \sqrt{2\lambda t^2 + 6\lambda t + 2\lambda + 1}\ \forall\, t > 0$ and $\forall\, \lambda \geq \frac{2}{5}$ and

then : $f''(t) = \frac{\lambda(5\lambda - 2)}{(2\lambda t^2 + 6\lambda t + 2\lambda + 1)^{\frac{3}{2}}} \overset{\lambda \geq \frac{2}{5}}{\geq} 0 \Rightarrow f(t)$ is convex $\therefore$ via

weighted Jensen's inequality, $\frac{y.f(t_1) + z.f(t_2) + x.f(t_3)}{x + y + z} \geq f\left(\frac{yt_1 + zt_2 + xt_3}{x + y + z} \right)$

$$\Rightarrow y.f(t_1) + z.f(t_2) + x.f(t_3) \geq f\left(\frac{y.\frac{x}{y} + z.\frac{y}{z} + x.\frac{z}{x}}{x + y + z} \right) (\because x + y + z = 1)$$

$$\Rightarrow \text{LHS of } (*) \geq f(1) = \frac{(1+1)}{2}.\sqrt{10\lambda + 1} - \sqrt{2\lambda.1^2 + 6\lambda.1 + 2\lambda + 1} = 0$$

$$\Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \sqrt{2\lambda x^2 + 6\lambda xy + (2\lambda + 1)y^2} \leq \sum_{cyc} \left(\left(\frac{x+y}{2} \right) . \sqrt{10\lambda + 1} \right)$$

$$= \sqrt{10\lambda + 1}.\frac{2(x + y + z)}{2} \overset{x+y+z=1}{=} \sqrt{10\lambda + 1}, \text{i.e.}, \sum_{cyc} \sqrt{2\lambda x^2 + 6\lambda xy + (2\lambda + 1)y^2}$$

$$\leq \sqrt{10\lambda + 1}\ \forall\, x, y, z > 0 \,|\, x + y + z = 1 \text{ and } \lambda \geq \frac{2}{5}, '' ='' \text{ iff } x = y = z = \frac{1}{3} \text{ (QED)}$$