ROMANIAN MATHEMATICAL MAGAZINE

If $0 < a \leq b \leq c \leq \frac{1}{e}$ then: $a^{a\left(b^b-c^c\right)}.b^{b\left(c^c-a^a\right)}.c^{c\left(a^a-b^b\right)} \leq 1$

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Let $x = a^a,\ \ y = b^b,\ \ z = c^c$ and $f(x) = x^x,\ \ x \in \left(0, \frac{1}{e}\right].$

We have $f'(x) = ln(xe).x^x \leq 0,$

then f is decreasing on $\left(0, \frac{1}{e}\right]$ and $0 < z \leq y \leq x \leq \lim_{x\to 0^+} f(x) = 1.$

The problem becomes to prove that

$$x^{y-z}.y^{z-x}.z^{x-y} \leq 1 \ \Leftrightarrow \ (y-z).lnx + (z-x).lny + (x-y).lnz \leq 0,$$

$$\Leftrightarrow \ (y-z)(lnx - lny) \leq (x-y)(lny - lnz) \ \Leftrightarrow \ (y-z).ln\left(\frac{x}{y}\right) \leq (x-y).ln\left(\frac{y}{z}\right),$$

Using the known inequality, $1 - \frac{1}{t} \leq lnt \leq t - 1,\ \ \forall t > 0,$ we have

$$(y-z).ln\left(\frac{x}{y}\right) \leq (y-z).\left(\frac{x}{y} - 1\right) = (x-y)\left(1 - \frac{z}{y}\right) \leq (x-y).ln\left(\frac{y}{z}\right),$$

which completes the proof. Equality holds iff $a = b = c$.