ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } \begin{cases} a, b > 0 \\ n \in \{2, 3, \dots\} \end{cases}, \text{then}:$$

$$1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \cdots + \frac{n(na+b)}{a+nb} > \frac{n(n+1)}{2(2^2.3^3.\dots.n^n)^{\frac{2}{n(n+1)}}}$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

$$(2^2.3^3.\dots.n^n)^{\frac{2}{n(n+1)}} = (1^1.2^2.3^3.\dots.n^n)^{\frac{1}{1+2+3+\cdots+n}} \overset{\text{Weighted GM > Weighted HM}}{>}$$

$$\frac{1+2+3+\cdots+n}{\frac{1}{1}+\frac{2}{2}+\frac{3}{3}+\cdots+\frac{n}{n}} \therefore (2^2.3^3.\dots.n^n)^{\frac{2}{n(n+1)}} > \frac{n(n+1)}{2n} \to (1)$$

$$\text{Also, } 1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \cdots + \frac{n(na+b)}{a+nb}$$

$$= 1 + \frac{a+2b+3a}{a+2b} + \frac{a+3b+8a}{a+3b} + \cdots + \frac{a+nb+(n^2-1)a}{a+nb}$$

$$= \underbrace{1+1+1+\cdots+1}_{n \text{ terms}} + \sum_{k=2}^{n} \frac{(k^2-1)a}{a+kb} = n + \sum_{k=2}^{n} \frac{(k^2-1)a}{a+kb}$$

$$\therefore 1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \cdots + \frac{n(na+b)}{a+nb} > n \to (2)$$

$$\therefore \left((2^2.3^3.\dots.n^n)^{\frac{2}{n(n+1)}}\right)\left(1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \cdots + \frac{n(na+b)}{a+nb}\right)$$

$$\overset{\text{via (1) and (2)}}{>} \frac{n(n+1)}{2n}.n = \frac{n(n+1)}{2}$$

$$\Rightarrow 1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \cdots + \frac{n(na+b)}{a+nb} > \frac{n(n+1)}{2(2^2.3^3.\dots.n^n)^{\frac{2}{n(n+1)}}}$$

$$\forall \begin{cases} a, b > 0 \\ n \in \{2, 3, \dots\} \end{cases} \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \cdots + \frac{n(na+b)}{a+nb} =$$

$$= 1 + \left(1 + \frac{3a}{a+2b}\right) + \left(1 + \frac{8a}{a+3b}\right) + \cdots + \left(1 + \frac{(n^2-1)a}{a+nb}\right) > 1+1+1+\cdots+1 = n.$$

So it suffices to prove that

$$\left(2^2.3^3\dots n^n\right)^{\frac{2}{n(n+1)}} \geq \frac{n+1}{2} \text{ or } 2.ln(2)+3.ln(3)+\cdots+n.ln(n) \geq \frac{n(n+1)}{2}.ln\left(\frac{n+1}{2}\right).$$

The function $f(x) = x.ln(x)$, $x > 0$, **is convex on** $(0,\infty)$, **then by Jensen's inequality, we have**

$$2ln(2)+3ln(3)+\cdots+nln(n) = \sum_{k=1}^{n} f(k) \geq nf\left(\frac{1}{n}\sum_{k=1}^{n} k\right) = nf\left(\frac{n+1}{2}\right) = \frac{n(n+1)}{2}.ln\left(\frac{n+1}{2}\right).$$

So the proof is complete.