ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ *then prove that* :

$$\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}\right).\sqrt[3]{a^3+2b^3+c^3} \le \sqrt{2}\left(\sqrt[3]{a^3+b^3}^{\,2}+\sqrt[3]{b^3+c^3}^{\,2}\right)$$

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$$\text{Power}-\text{Mean inequality} \Rightarrow \left(\frac{a^3+b^3}{2}\right)^{\frac{1}{3}} \ge \left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}$$

$$\Rightarrow \sqrt{\frac{a^2+b^2}{2}} \le \left(\frac{a^3+b^3}{2}\right)^{\frac{1}{3}} \text{ and similarly, } \sqrt{\frac{b^2+c^2}{2}} \le \left(\frac{b^3+c^3}{2}\right)^{\frac{1}{3}}$$

$$\Rightarrow \left(\sqrt{\frac{a^2+b^2}{2}}+\sqrt{\frac{b^2+c^2}{2}}\right).\sqrt[3]{(a^3+b^3)+(b^3+c^3)}$$

$$\le (x+y).\sqrt[3]{2x^3+2y^3}\ \left(x=\left(\frac{a^3+b^3}{2}\right)^{\frac{1}{3}}, y=\left(\frac{b^3+c^3}{2}\right)^{\frac{1}{3}}\right)$$

$$\overset{?}{\le} \sqrt[3]{a^3+b^3}^{\,2}+\sqrt[3]{b^3+c^3}^{\,2} = 2^{\frac{2}{3}}.(x^2+y^2) \Leftrightarrow 4(x^2+y^2)^3 \overset{?}{\ge} 2(x^3+y^3)(x+y)^3$$

$$\Leftrightarrow 2(t^2+1)^3 \overset{?}{\ge} (t^3+1)(t+1)^3\ \left(t=\frac{x}{y}\right)$$

$$\Leftrightarrow t^6-3t^5+3t^4-2t^3+3t^2-3t+1 \overset{?}{\ge} 0 \Leftrightarrow (t^2+t+1)(t-1)^4 \overset{?}{\ge} 0 \rightarrow \text{true}$$

$$\therefore \left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}\right).\sqrt[3]{a^3+2b^3+c^3} \le \sqrt{2}\left(\sqrt[3]{a^3+b^3}^{\,2}+\sqrt[3]{b^3+c^3}^{\,2}\right)$$

$$\forall a, b, c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$