ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, $4\left(\frac{2}{a}+b^2\right) \geq 7+5e^{\left(2c+\frac{1}{a^2}-3\right)}$ and

$27\left(\frac{a}{2c+a}+\frac{b}{2a+b}+\frac{c}{2b+c}\right)=2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+21$, then $a, b, c = ?$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+21 \overset{?}{\geq} 27\left(\frac{a}{2c+a}+\frac{b}{2a+b}+\frac{c}{2b+c}\right)$$

$$\Leftrightarrow 2\sum_{cyc}\frac{a}{b}+21 \overset{?}{\geq} 27\sum_{cyc}\frac{b+2a-2a}{2a+b} \Leftrightarrow \sum_{cyc}\frac{a}{b}+27\sum_{cyc}\frac{a}{2a+b} \overset{?}{\geq} 30$$

$$\Leftrightarrow \sum_{cyc}\frac{a}{b}+27\sum_{cyc}\frac{\frac{a}{b}}{2.\frac{a}{b}+1} \overset{?}{\geq} 30$$

$$\Leftrightarrow \sum_{cyc}x+27\sum_{cyc}\frac{x}{2x+1} \overset{?}{\geq} 30 \ \left(x=\frac{a}{b}, y=\frac{b}{c}, z=\frac{c}{a}\right)$$

$$\Leftrightarrow 4xyz\sum_{cyc}x+2\sum_{cyc}(x^2y+xy^2)+48xyz+\sum_{cyc}x^2 \overset{?}{\geq} 4\sum_{cyc}xy+16\sum_{cyc}x+15$$

$$\Leftrightarrow 4xyz\sum_{cyc}x+2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-6xyz+48xyz+\sum_{cyc}x^2 \overset{?}{\geq} 4\sum_{cyc}xy+16\sum_{cyc}x+15$$

$$\Leftrightarrow 2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)+\sum_{cyc}x^2+27 \overset{?}{\geq} 4\sum_{cyc}xy+12\sum_{cyc}x \ (\because xyz=1)$$

$$\Leftrightarrow 2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x-2\right)+\sum_{cyc}x^2+27 \overset{?}{\geq} 12\sum_{cyc}x$$

$$\Leftrightarrow \left(\left(\sum_{cyc}x\right)^2-\sum_{cyc}x^2\right)\left(\sum_{cyc}x-2\right)+\sum_{cyc}x^2+27 \overset{?}{\geq} 12\sum_{cyc}x$$

$$\Leftrightarrow \left(\sum_{cyc}x\right)^2\left(\sum_{cyc}x-2\right)+27 \underset{(*)}{\overset{?}{\geq}} 12\sum_{cyc}x+\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x-3\right)$$

$$\text{Now, } \left(\sum_{cyc}xy\right)^2 \geq 3xyz\left(\sum_{cyc}x\right) \overset{xyz=1}{=} 3\sum_{cyc}x \Rightarrow \sum_{cyc}xy \geq \sqrt{3\sum_{cyc}x}$$

$$\therefore \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \le \left(\sum_{cyc} x\right)^2 - 2.\sqrt{3\sum_{cyc} x} \text{ and}$$

$$\because \sum_{cyc} x - 3 = \sum_{cyc} \frac{a}{b} - 3 \overset{A-G}{\ge} 0, \therefore 12\sum_{cyc} x + \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x - 3\right)$$

$$\le 12\sum_{cyc} x + \left(\left(\sum_{cyc} x\right)^2 - 2.\sqrt{3\sum_{cyc} x}\right)\left(\sum_{cyc} x - 3\right) \overset{?}{\le} \left(\sum_{cyc} x\right)^2\left(\sum_{cyc} x - 2\right) + 27$$

$$\Leftrightarrow 27 + \frac{p^4}{9}\left(\frac{p^2}{3} - 2\right) \overset{?}{\ge} 12.\frac{p^2}{3} + \left(\frac{p^4}{9} - 2p\right)\left(\frac{p^2}{3} - 3\right)\left(\text{where } p = \sqrt{3\sum_{cyc} x}\right)$$

$$\Leftrightarrow \frac{p^4 + 6p^3 - 12p^2 - 18p + 81}{27} \overset{?}{\ge} 0 \Leftrightarrow \frac{(p-3)^2(p+3)(p+9)}{27} \overset{?}{\ge} 0 \to \text{true}$$

$$\because p = \sqrt{3\sum_{cyc} x} = \sqrt{3\sum_{cyc} \frac{a}{b}} \overset{A-G}{\ge} 3 \Rightarrow (*) \text{ is true}$$

$$\therefore \boxed{2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + 21 \ge 27\left(\frac{a}{2c+a} + \frac{b}{2a+b} + \frac{c}{2b+c}\right), ''='' \text{ iff } x = y = z \text{ and } \sqrt{3\sum_{cyc} \frac{a}{b}} = 3}$$

$$\to (1) \text{ Now, } \sqrt{3\sum_{cyc} \frac{a}{b}} \overset{A-G}{\ge} 3\ (''='' \text{ iff } a = b = c) \Rightarrow \sqrt{3\sum_{cyc} \frac{a}{b}} = 3 \text{ iff } a = b = c$$

$$\text{and also } x = y = z \text{ iff } \frac{a}{b} = \frac{b}{c} = \frac{c}{a} \Rightarrow \text{iff } a = b = c \to (2)$$

$$\text{So, (1) and (2)} \Rightarrow 27\left(\frac{a}{2c+a} + \frac{b}{2a+b} + \frac{c}{2b+c}\right) = 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + 21$$

$$\Rightarrow \boxed{\boxed{a = b = c}}$$

$$\text{Let } f(m) = e^m - 1 - m - \frac{m^2}{2}\ \forall\, m \ge 0 \text{ and then : } f'(m) = e^m - 1 - m \ge 0$$

$$\Rightarrow f(m) \text{ is} \uparrow \text{on } [0, \infty) \Rightarrow f(m) \ge f(0) = 1 - 1 \therefore e^m \ge 1 + m + \frac{m^2}{2}\ \forall\, m \ge 0 \to (\bullet)$$

$$\text{Via } (\bullet), 7 + 5e^{\left(2a + \frac{1}{a^2} - 3\right)} - 4\left(\frac{2}{a} + a^2\right) \ge$$

$$7 + 5\left(1 + \left(2a + \frac{1}{a^2} - 3\right) + \left(2a + \frac{1}{a^2} - 3\right)^2\right) - 4\left(\frac{2}{a} + a^2\right)$$

$$= \frac{16a^6 - 50a^5 + 42a^4 + 12a^3 - 25a^2 + 5}{a^4}$$

$$= \frac{(a-1)^2\left((16a^2 + 14a + 2)(a-1)^2 + 3\right)}{a^4} \ge 0 \because a > 0$$

$$\therefore \boxed{4\left(\frac{2}{a} + a^2\right) \le 7 + 5e^{\left(2a + \frac{1}{a^2} - 3\right)}}, ''='' \text{ iff } 2a + \frac{1}{a^2} - 3 = 0$$

$$\Rightarrow \text{iff } \frac{(2a+1)(a-1)^2}{a^2} = 0 \Rightarrow \boxed{''='' \text{ iff } a = 1} \to (3)$$

$$\text{Now}, \because a = \mathrm{b} = \mathrm{c} \therefore 4\left(\frac{2}{a} + \mathrm{b}^2\right) \geq 7 + 5\mathrm{e}^{\left(2\mathrm{c}+\frac{1}{a^2}-3\right)} \Rightarrow$$

$$4\left(\frac{2}{a} + a^2\right) \geq 7 + 5\mathrm{e}^{\left(2a+\frac{1}{a^2}-3\right)}, \text{but via } (3), 4\left(\frac{2}{a} + a^2\right) \leq 7 + 5\mathrm{e}^{\left(2a+\frac{1}{a^2}-3\right)}$$

$$\therefore 4\left(\frac{2}{a} + a^2\right) = 7 + 5\mathrm{e}^{\left(2a+\frac{1}{a^2}-3\right)} \text{ and via } (3), \boxed{a = 1} \therefore a = \mathrm{b} = \mathrm{c} = 1 \; (ans)$$