ROMANIAN MATHEMATICAL MAGAZINE

$$\left.\begin{matrix}\text{If } x>0\\ n\in\{1,2,3,\dots\}\end{matrix}\right\}, \text{then the following relationship holds:}$$

$$\left(\frac{x}{x+1}\right)^{x}+2\left(\frac{x+1}{x+2}\right)^{x+1}+\cdots+n\left(\frac{x+n}{x+n+1}\right)^{x+n}>\frac{n(n+1)}{2e}$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } f(m)=\ln m-1+\frac{1}{m}\ \forall\, m>0 \text{ and then}: f'(m)=\frac{m-1}{m^2}$$

$$\boxed{Case\ 1}\ m\le 1 \text{ and then}: f'(m)\le 0\ \forall\, m\le 1\Rightarrow f(m) \text{ is} \downarrow \forall\, m\in(0,1]$$

$$\Rightarrow f(m)\ge f(1)=0$$

$$\boxed{Case\ 2}\ m\ge 1 \text{ and then}: f'(m)\ge 0\ \forall\, m\ge 1\Rightarrow f(m) \text{ is} \uparrow \forall\, n\in[1,\infty)$$

$$\Rightarrow f(m)\ge f(1)=0 \therefore \text{combining both cases}, \ln m\ge 1-\frac{1}{m}\ \forall\, m>0, ''='' \text{ iff } m=1$$

$$(x+n)\ln\left(\frac{x+n}{x+n+1}\right)>(x+n)\left(1-\frac{x+n+1}{x+n}\right)$$

$$\left(\text{strict inequality} \because \frac{x+n}{x+n+1}\ne 1\right)\Rightarrow \ln\left(\frac{x+n}{x+n+1}\right)^{x+n}>-1$$

$$\Rightarrow\left(\frac{x+n}{x+n+1}\right)^{x+n}>\frac{1}{e}\ \forall\, x>0 \text{ and } \forall\, n\in\{0,1,2,3,\dots\}$$

$$\therefore \left(\frac{x}{x+1}\right)^{x},\left(\frac{x+1}{x+2}\right)^{x+1},\dots,\left(\frac{x+n}{x+n+1}\right)^{x+n}>\frac{1}{e}$$

$$\Rightarrow\left(\frac{x}{x+1}\right)^{x}+2\left(\frac{x+1}{x+2}\right)^{x+1}+\cdots+n\left(\frac{x+n}{x+n+1}\right)^{x+n}>\frac{1}{e}+\frac{2}{e}+\cdots+\frac{n}{e}=\frac{n(n+1)}{2e}$$

(QED)