ROMANIAN MATHEMATICAL MAGAZINE

**If $n \in \{3,4,5,\dots\}$, then the following relationship holds:**

$$n^n < (n!)^2 < n^n.\left(\frac{n+1}{3}\right)^{n-1}$$

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$$\text{Via Stirling's approximation, } n^n < (n!)^2 < n^n.\left(\frac{n+1}{3}\right)^{n-1}$$

$$\Leftrightarrow n^n < 2\pi n\left(\frac{n}{e}\right)^{2n} < n^n.\left(\frac{n+1}{3}\right)^{n-1}$$

$$\text{Now, } n^n < 2\pi n\left(\frac{n}{e}\right)^{2n} \Leftrightarrow 2\pi n\left(\frac{n}{e^2}\right)^{n} > 1 \Leftrightarrow \ln(2\pi) + \ln(n) + n(\ln(n) - 2) > 1$$

$$\Leftrightarrow \ln(2\pi) + (n+1)\ln(n) - 2n \overset{(i)}{>} 0$$

Let $f(n) = \ln(2\pi) + (n+1)\ln(n) - 2n$ be a continuous function and then:

$$f'(n) = \ln(n) + \frac{1}{n} - 1 \text{ and } f''(n) = \frac{n-1}{n^2}$$

$\boxed{\text{Case 1}}$ $n \le 1$ and then: $f''(n) \le 0\ \forall\ n \le 1 \Rightarrow f'(n)$ is $\downarrow\ \forall\ n \in (0,1]$
$\Rightarrow f'(n) \ge f'(1) = 0$

$\boxed{\text{Case 2}}$ $n \ge 1$ and then: $f''(n) \ge 0\ \forall\ n \ge 1 \Rightarrow f'(n)$ is $\uparrow\ \forall\ n \in [1,\infty)$,
$\Rightarrow f'(n) \ge f'(1) = 0 \therefore \forall\ n > 0, f'(n) \ge 0$ ('' = '' iff $n = 1$) $\Rightarrow \forall\ n \ge 3, f'(n) > 0$
$\Rightarrow \forall\ n \ge 3, f(n) \ge f(3) = \ln(2\pi) + 4\ln(3) - 6 \approx 0.2323 > 0 \Rightarrow$ (i) is true

$$\Rightarrow \forall\ n \ge 3, n^n < 2\pi n\left(\frac{n}{e}\right)^{2n} \Rightarrow \forall\ n \in \{3,4,5,\dots\}, \boxed{(n!)^2 > n^n}$$

$$\text{Again, } 2\pi n\left(\frac{n}{e}\right)^{2n} < n^n.\left(\frac{n+1}{3}\right)^{n-1} \Leftrightarrow 2\pi n\left(\frac{n}{e^2}\right)^{n} < \left(\frac{n+1}{3}\right)^{n-1}$$

$$\Leftrightarrow \ln(2\pi) + \ln(n) + n\ln(n) - 2n < (n-1)\ln\left(\frac{n+1}{3}\right)$$

$$\Leftrightarrow (n-1)\ln\left(\frac{n+1}{3}\right) - \ln(2\pi) - (n+1)\ln(n) + 2n \overset{(ii)}{>} 0$$

Let $F(n) = (n-1)\ln\left(\frac{n+1}{3}\right) - \ln(2\pi) - (n+1)\ln(n) + 2n$ be a continuous function and then: $F'(n) = \ln\left(\frac{n+1}{3}\right) - \ln(n) + \frac{n-1}{n+1} - \frac{1}{n} + 1$ and

$$F''(n) = \frac{2n^2+n+1}{n^2(n+1)^2} > 0\ \forall\ n \ge 3 \Rightarrow F'(n) \text{ is } \uparrow\ \forall\ n \ge 3$$

$$\Rightarrow F'(n) \ge F'(3) \approx 0.3557 > 0 \Rightarrow F(n) \text{ is } \uparrow\ \forall\ n \ge 3 \Rightarrow F(n) \ge F(3) \approx 0.343 > 0$$

$$\Rightarrow \text{(ii) is true} \Rightarrow \forall\ n \ge 3, 2\pi n\left(\frac{n}{e}\right)^{2n} < n^n.\left(\frac{n+1}{3}\right)^{n-1} \Rightarrow \forall\ n \in \{3,4,5,\dots\},$$

$$\boxed{(n!)^2 < n^n.\left(\frac{n+1}{3}\right)^{n-1}} \therefore n^n < (n!)^2 < n^n.\left(\frac{n+1}{3}\right)^{n-1}\ \forall\ n \in \{3,4,5,\dots\} \text{ (QED)}$$