If $a \geq b \geq c > 0$, then :

$$108\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \leq 189+5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)^3$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India, Solutions 2,3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=\frac{b^2-a^2}{ab}+\frac{c^2-b^2}{bc}+\frac{a^2-c^2}{ac}$$

$$=\frac{c(b+a)(b-a)+c^2(a-b)+ab(a-b)}{abc}=\frac{(b-a)\left(bc+ac-c^2-ab\right)}{abc}$$

$$=\frac{(b-a)\left(b(c-a)-c(c-a)\right)}{abc}=\frac{(b-a)(c-a)(b-c)}{abc}=\frac{(a-b)(b-c)(a-c)}{abc}$$

$$\geq 0\ (\because a\geq b\geq c>0) \Rightarrow x+y+z\geq \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ \left(x=\frac{b}{a}, y=\frac{c}{b}, z=\frac{a}{c}\right)$$

$$\Rightarrow \sum_{cyc} x \geq \sum_{cyc} xy \rightarrow (1)\ (\because xyz=1) \text{ and also}, \left(\sum_{cyc} xy\right)^2 \geq 3xyz\left(\sum_{cyc} x\right)$$

$$\Rightarrow \sum_{cyc} xy \geq \sqrt{3\sum_{cyc} x} \rightarrow (2)\ (\because xyz=1)$$

$$\text{Now, } 189+5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)^3-108\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=189+5\left(\sum_{cyc} x\right)^3-108\sum_{cyc} xy$$

$$\overset{\text{via }(1)}{\geq} 189+5\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^2-108\sum_{cyc} xy$$

$$=189+\left(\sum_{cyc} xy\right)\left(5\left(\sum_{cyc} x\right)^2-45-63\right) \overset{\text{via }(1)\text{ and }(2)}{\geq}$$

$$189+\sqrt{3\sum_{cyc} x}\cdot\left(5\left(\sum_{cyc} x\right)^2-45\right)-63\left(\sum_{cyc} x\right)$$

$$\left(\because \sum_{cyc} x \overset{A-G}{\geq} 3\sqrt[3]{xyz} \overset{xyz=1}{=} 3 \Rightarrow 5\left(\sum_{cyc} x\right)^2-45\geq 0\right)$$

$$= 189 + t.\left(\frac{5t^4}{9} - 45\right) - 63.\frac{t^2}{3}\left(\text{where } t = \sqrt{3\sum_{cyc} x}\right)$$

$$= \frac{5t^5 - 189t^2 - 405t + 1701}{9}$$

$$= \frac{(t-3)\left((t-3)(5t^3 + 30t^2 + 135t + 351) + 486\right)}{9} \geq 0$$

$$\left(\because \sum_{cyc} x \overset{A-G}{\geq} 3\sqrt[3]{xyz} \overset{xyz=1}{=} 3 \Rightarrow t = \sqrt{3\sum_{cyc} x} \geq \sqrt{3.3} \Rightarrow t \geq 3\right)$$

$$\therefore 108\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leq 189 + 5\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)^3$$

$$\forall a \geq b \geq c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x := \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$. By AM − GM inequality, we have $x \geq 3$.

Since $\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) - \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) = \frac{(a-b)(a-c)(b-c)}{abc} \geq 0$,

then $\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq x$, and

$$189 + 5\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)^3 \geq 189 + 5x^3 = 108x + (x-3)(5x^2 + 15x - 63)$$

$$\overset{x \geq 3}{\geq} 108x = 108\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right),$$

which completes the proof. Equality holds iff $a = b = c$.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x := \frac{b}{a}$, $y := \frac{c}{b}$, $z := \frac{a}{c}$ and $p := x + y + z$, $q := xy + yz + zx$, $r := xyz = 1$.

The desired inequality is equivalent to

$$108q \leq 189 + 5p^3.$$

By AM − GM inequality, we have $p \geq 3\sqrt[3]{r} = 3$.

Also by Schur's inequality, we have $4pq \leq p^3 + 9r$, then

$$108q \leq \frac{27(p^3 + 9)}{p} = 189 + 5p^3 - \frac{(p-3)^2(5p^2 + 3p - 27)}{p} \overset{p \geq 3}{\leq} 189 + 5p^3,$$

which completes the proof. Equality holds iff $a = b = c$.