ROMANIAN MATHEMATICAL MAGAZINE

If $n \in \mathbb{N}^* - \{1\}$, then the following relationship holds :

$$\binom{2n}{n} \ge 1 + n^2 + \binom{2n}{n-2}$$

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$$\text{If } n = 2, \binom{2n}{n} - \left(1 + n^2 + \binom{2n}{n-2}\right) = \binom{4}{2} - \left(1 + 4 + \binom{4}{0}\right) = 0$$

$$\Rightarrow \binom{2n}{n} = 1 + n^2 + \binom{2n}{n-2} \text{ for } n = 2$$

$$\text{If } n = 3, \binom{2n}{n} - \left(1 + n^2 + \binom{2n}{n-2}\right) = \binom{6}{3} - \left(1 + 9 + \binom{6}{1}\right) = 20 - 10 - 6 = 4$$

$$\Rightarrow \binom{2n}{n} > 1 + n^2 + \binom{2n}{n-2} \text{ for } n = 3$$

$$\text{If } n = 4, \binom{2n}{n} - \left(1 + n^2 + \binom{2n}{n-2}\right) = \binom{8}{4} - \left(1 + 16 + \binom{8}{2}\right) = 70 - 17 - 28$$

$$= 25 \Rightarrow \binom{2n}{n} > 1 + n^2 + \binom{2n}{n-2} \text{ for } n = 4$$

$$\text{If } n = 5, \binom{2n}{n} - \left(1 + n^2 + \binom{2n}{n-2}\right) = \binom{10}{5} - \left(1 + 25 + \binom{10}{3}\right)$$

$$= 252 - 26 - 120 = 106 \Rightarrow \binom{2n}{n} > 1 + n^2 + \binom{2n}{n-2} \text{ for } n = 5$$

$$\text{If } n = 6, \binom{2n}{n} - \left(1 + n^2 + \binom{2n}{n-2}\right) = \binom{12}{6} - \left(1 + 36 + \binom{12}{4}\right)$$

$$= 924 - 37 - 495 > 0 \Rightarrow \binom{2n}{n} > 1 + n^2 + \binom{2n}{n-2} \text{ for } n = 6$$

$$\text{If } n = 7, \binom{2n}{n} - \left(1 + n^2 + \binom{2n}{n-2}\right) = \binom{14}{7} - \left(1 + 49 + \binom{14}{5}\right)$$

$$= 3432 - 50 - 2002 > 0 \Rightarrow \binom{2n}{n} > 1 + n^2 + \binom{2n}{n-2} \text{ for } n = 7$$

$$\text{We now consider } n \ge 8; n \in \mathbb{N}^* \text{ and then : } \boxed{\binom{2n}{n} - \binom{2n}{n-2}}$$

$$= \frac{1.2.3 \dots n.(n+1)(n+2)(n+3)(n+4)\dots 2n}{(1.2.3 \dots n)(1.2.3 \dots n)} - \frac{1.2.3 \dots n.(n+1)(n+2)(n+3)(n+4)\dots 2n}{\big(1.2.3 \dots (n-2)\big)\big(1.2.3 \dots n(n+1)(n+2)\big)}$$

$$= \frac{(n+1)(n+2)(n+3)(n+4)\dots 2n}{1.2.3 \dots (n-2)(n-1)n} - \frac{(n+3)(n+4)\dots 2n}{1.2.3 \dots (n-2)}$$

$$= \frac{(n+3)(n+4)\dots 2n}{1.2.3 \dots (n-2)} \cdot \left(\frac{(n+1)(n+2)}{n(n-1)} - 1\right) = \frac{(n+3)(n+4)\dots 2n}{1.2.3 \dots (n-2)} \cdot \frac{4n+2}{n^2-n}$$

$$= \frac{2n}{n-2} \cdot \frac{n+3}{1} \cdot \frac{n+4}{2} \cdot \frac{n+5}{3} \cdot \frac{4n+2}{n^2-n} \cdot \frac{(n+6)(n+7)\dots(2n-1)}{4.5 \dots (n-3)}$$

$$=\boxed{\frac{2n}{n-2}\cdot\frac{n+3}{1}\cdot\frac{n+4}{2}\cdot\frac{n+5}{3}\cdot\frac{4n+2}{n^2-n}\cdot\prod_{k=6}^{n-1}\frac{n+k}{k-2}}$$

$$>\frac{2n}{n-2}\cdot\frac{n+3}{1}\cdot\frac{n+4}{2}\cdot\frac{n+5}{3}\cdot\frac{4n+2}{n^2-n}\left(\because\frac{n+k}{k-2}>1\ \forall\ k\in\{6.7\ldots(n-1)\}\right)$$

$$\boxed{\overset{?}{>}1+n^2}\Leftrightarrow\boxed{\boxed{n^5+59n^4+203n^3+343n^2+114n\overset{?}{>}0}}\rightarrow \text{true}$$

$$\therefore\binom{2n}{n}>1+n^2+\binom{2n}{n-2}\ \text{for } n\geq 8; n\in\mathbb{N}^*\ \therefore \text{combining } \textit{all cases},$$

$$\binom{2n}{n}\geq 1+n^2+\binom{2n}{n-2}\ \forall\ n\in\mathbb{N}^*-\{1\},''=''\ \text{iff } n=2\ (\text{QED})$$