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If $n \in \{2, 3, \dots\}$, then prove that

$$1 + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \cdot \sqrt[3]{\frac{n(n+1)^2}{4}} > 2\sqrt[n]{n!}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \geq \frac{n^2}{1 + 2 + \cdots + n} = \frac{2n}{n+1}$$

Then

$$1 + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \cdot \sqrt[3]{\frac{n(n+1)^2}{4}} \geq 1 + \frac{2n}{n+1} \cdot \sqrt[3]{\frac{n(n+1)^2}{4}}$$

$$= 1 + n \cdot \sqrt[3]{\frac{2n}{n+1}} \overset{n>1}{>} 1 + n =$$

$$= 2 \cdot \frac{1 + 2 + \cdots + n}{n} \overset{AM-GM}{>} 2 \cdot \sqrt[n]{1 \cdot 2 \dots n} = 2 \cdot \sqrt[n]{n!}.$$